First Annual Columbus State Calculus Tournament

Sponsored by
The Columbus State University
Department of Mathematics
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The Columbus State University Mathematics faculty welcome you to this year's Pre-Calculus/Calculus tournament. We wish you success on this test and in your future studies.

Instructions

This is a 120-minute, 10-problem or 20-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

The examination will be scored on the basis of $+12$ *for each correct answer,* -3 *for each incorrect selection, and* $+1$ *for each omitted item.*

No phones or any communication devices can be used. Calculators with CAS such as the TI-89 are not allowed. In fact, the test is designed in such a way that you do not really need a calculator. The problems denoted with $[\star^n]$ are the tie-breaker problems, so more attention should be given to those in the spare time and please add written justification for your answers on extra paper for these problems in the order $[\star^1]$, $[\star^2]$, etc. It is not really necessary, but you may find useful to read the "Theoretical facts" part.

Do not open your test until instructed to do so!

Theoretical facts that you may find useful.

In this exam:

(1) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(2) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

(3) The derivative of a function f is denoted by f'. Second and third derivatives are denoted by f'' and f''' respectively.

Theorem 1: If a function, which is two times differentiable, satisfies $f''(x) \geq 0$ for x in some interval I, we can conclude that f is convex upward on I.

Theorem 2: A differentiable function for which $f'(x)$ changes sign at a, then f has a local extrema at a.

Theorem 3: The area of the region between the graphs of f and g (continuous functions), $x \in [a, b]$, is given by

$$A = \int_a^b |f(x) - g(x)| dx$$

Theorem 4: The fundamental theorem of calculus:

(a) If f is continuous on the interval $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is differentiable and $F'(x) = f(x)$ for all $x \in [a, b]$.

(b) If f Riemann integrable on $[a, b]$ and F is an antiderivative of f, then $\int_a^b f(t)dt = F(b) - F(a)$.

Theorem 5: If f is differentiable and $f'(x) \geq 0$ on some interval, then f is non-decreasing on that interval.

Definition: A function is Riemann integrable on $[a, b]$ if the following limit exists and it is independent of the partition $\Delta = (x_0 = a, x_1, ..., x_{n-1}, x_n = b)$, and the points $c_i \in [x_{i-1}, x_i]$, $i = 1, 2, ..., n$:

$$\lim_{||\Delta|| \to 0} \sum f(c_i)(x_i - x_{i-1}) = L.$$

If this is the case, L is written as $\int_a^b f(t)dt$.

Differentiation Rules: $(fg)' = f'g + fg'$, $(f \circ g)' = (f' \circ g)g'$, $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$ for all $|x| < 1$, $\frac{d}{dx} x^\alpha = \alpha x^{\alpha-1}$ for all $x > 0$, $(f/g)' = (f'g - fg')/g^2$, $(u^v)' = vu^{v-1}u' + u^v(\ln u)v'$, $\frac{d}{dx} \tan x = \sec^2 x$.

(IVT) Intermediate Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous, and y is between $f(a)$ and $f(b)$ then there exits $c \in [a, b]$, such that $f(c) = y$.

Calculus Problems

1. (I) Every function f is continuous.

   (II) Every continuous function is differentiable.

   (III) Every differentiable function is continuous.

   (IV) There exist continuous functions which are not differentiable.

   Which of the above statements are true?

   (A) (I) and (II) (B) (I) and (III) (C) (II) and (III)

   (D) (III) and (IV) (E) (I) and (IV)

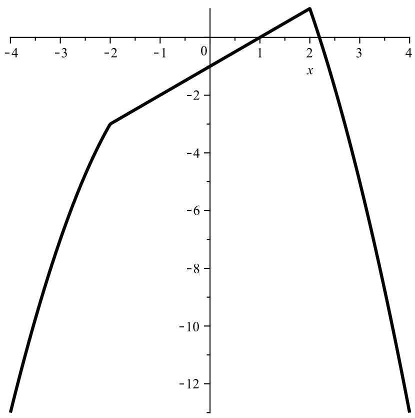
2. The Intermediate Value Theorem can be used for the function $f(x) = 2\sin x - \cos x$ on the interval $[a, b]$ to obtain a value $c \in (a, b)$ such that $f(c) = 0$. What is a good option for $[a, b]$?

   (A) $[\frac{\pi}{2}, \pi]$ (B) $[\frac{\pi}{6}, \frac{\pi}{4}]$ (C) $[-\frac{\pi}{2}, 0]$ (D) $[-\frac{\pi}{4}, 0]$ (E) $[0, \pi]$

3. The values of a and b are chosen in such a way the function

$$f(x) = \begin{cases} ax^2 + bx - 1 \text{ if } x \leq -2, \\ b - ax \text{ if } x \in (-2, 2), \\ bx^2 + ax + 7 \text{ if } x \geq 2, \end{cases}$$

   is continuous on the whole real line (graph as in the adjacent figure). What is $(a+b)^2$?



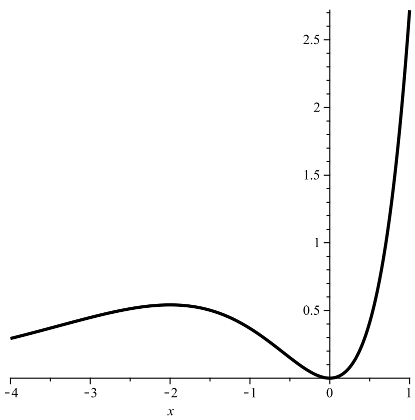
   (A) 4 (B) 3 (C) 2

   (D) 1 (E) 0

4. The function $f(x) = x^2e^x$ is convex downward on the interval $[a, b]$, convex upward on $(-\infty, a]$ and also on $[b, \infty)$. Then what is ab ?

(A) 1 (B) 2 (C) 3

(D) 4 (E) 5



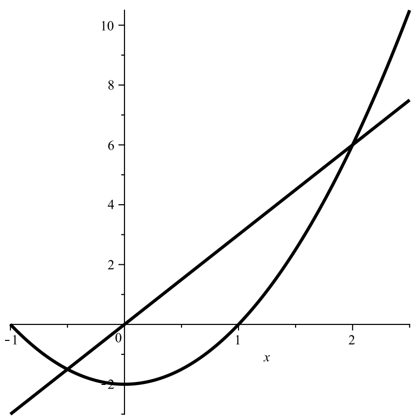
5. The function $f(x) = \dfrac{1-x}{x^2+2}$ has a maximum value of $f(a)$ and a minimum value of $f(b)$. Then, what is the value of $a^2 + b^2$?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

6. The area of the regions between the graphs of equations $y = 3x$ and $y = 2(x^2 - 1)$ is a rational number which can be written in reduced form as $\dfrac{5^m}{3(2^n)}$, for some integers m and n (see the adjacent figure). What is $m(n+1)$?

(A) 10 (B) 12 (C) 14

(D) 16 (E) 18



7. Suppose that f is differentiable and $g(x) = x^2f(1/x)$. If $f(1) = 3$ and $f'(1) = 4$, what is $g'(1)$?

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

8. What is the value of the limit

$$\lim_{x\to\infty} \frac{\int_0^x \sqrt{3+16t^2}dt}{x^2+1}.$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

9. The positive integers m and n are relatively prime, and chosen in such a way that

$$\lim_{x\to 0} \frac{3\sqrt{49+x}-7\sqrt{9-x}}{x} = \frac{m}{n}.$$

What is $m+n$?

(A) 20 (B) 30 (C) 40 (D) 50 (E) 60

10. We let m be the smallest positive integer such that

$$m\int_0^2 \frac{1-x}{\sqrt{1+4x}}dx$$

is also an integer. What is m?

(A) 5 (B) 6 (C) 3 (D) 4 (E) 0

11. Determine the equation of the tangent line to the graph of equation $(3x+2y)^4-x^2y=2$ at the point $(1,-1)$.

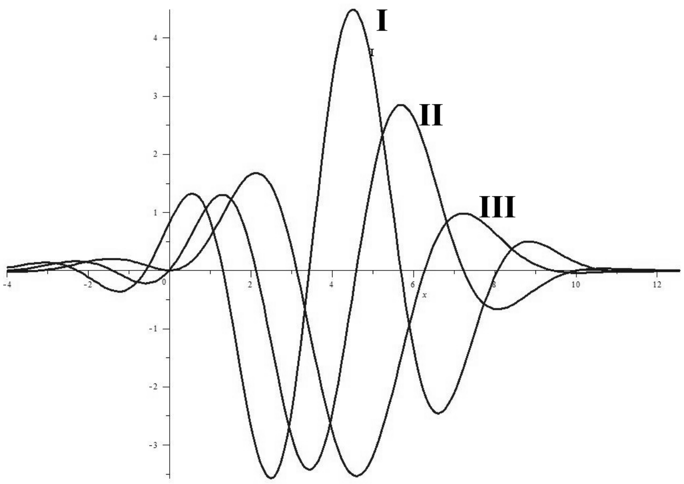
(A) $y=2x-3$ (B) $y=1-2x$ (C) $3y+4x=1$

(D) $y=-x$ (E) $y=x-2$

12. In the accompanying figure we have the graphs of f, f' and f''. Identify these graphs with the roman numerals shown.

(A) $\begin{matrix} I=f \\ II=f' \end{matrix}$ (B) $\begin{matrix} I=f \\ III=f' \end{matrix}$ (C) $\begin{matrix} II=f \\ III=f' \end{matrix}$

(D) $\begin{matrix} II=f \\ I=f' \end{matrix}$ (E) $\begin{matrix} III=f \\ II=f' \end{matrix}$



13. If $f(x) = x^{x^2}$, then its derivative satisfies

$$f'(x) = xf(x)(m + n \ln x), \text{ for all } x > 0,$$

with m and n some positive real numbers. What is $m + n$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

14. For m and n relatively prime positive integers, we have

$$\lim_{x \to 0} (\cos x)^{\cot^2 x} = e^{-m/n}.$$

What is $n - m$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

15. *Find $F'(1)$ for the function $F(x) = \int_{1/x^5}^{x^5} \frac{1}{1+t^4} dt$, $x \in (0, 2)$.*

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

16. *Let f be a continuous function defined on $[-1, 1]$ and such that $f(x) + f(-x) = x^4$ for all $x \in [-1, 1]$. What is $\int_{-1}^{1} f(x)dx$?*

(A) 1/2 (B) 1/3 (C) 1/4 (D) 1/5 (E) 1/6

17. [$*^4$] We let $A(1,1)$ and $B(3,81)$ be two points on the graph of $y = x^4$. We consider a point $C(c, c^4)$ on the same graph and in between A and B, such that the triangle ABC has the greatest area. What is then c^3?

(A) 8 (B) 10 (C) 12 (D) 14 (E) 16

18. [$*^3$] The function E satisfies the differential equation $E'(t) = E(t)^2$ and the initial condition $E(0) = 1$. What is the value of $E(3/4)$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

19. [$*^2$] Find the length of the curve given by $y = \arcsin(x) + \sqrt{1-x^2}$, $x \in [-1, 1]$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

20. [$*^1$] Find the limit

$$\lim_{n\to\infty} \frac{1}{n}(2^{1/n} + 2^{2/n} + \cdots + 2^{n/n}).$$

(A) $\sqrt[3]{2}$ (B) $1 + \frac{\ln 2}{2}$ (C) $2\ln 2$ (D) $\sqrt{2}$ (E) $\frac{1}{\ln 2}$