Notes for College Geometry, Fall 2015, E. Ionascu

1. Axiomatic Method

The axiomatic method originated with the ancient Greeks who have first put together in the Euclid's Elements a deductive way of arriving at the main results of elementary geometry from a small number of assumptions (postulates) considered self-evident.

1. Every axiomatic system has to have concepts that cannot be defined in terms of previously defined ones. Indeed, if one starts with concept C_1 which is defined in terms of C_2, and this goes on to say that in general C_n is defined in terms of C_{n+1}. Either this sequence is infinite and all these terms are distinct. Or, one of these terms repeats and we have a circular system of definitions. Usually, circular systems do not make much sense in mathematics or elsewhere. One can check that circular systems are very common by just opening a regular dictionary. If the sequence C_n is infinite we have some open theory which is probably difficult to understand as a whole. So, the alternative is that one has to have at least one concept undefined. These terms are usually called undefined terms.

Example: Most of the time concepts like "point", "line", "plane", are left undefined and most of the constructions of an axiomatic system for Euclidean Geometry. In some constructions terms like "incidence of point and line", "between", "congruence", "distance", "measure of an angle" are considered primary terms. (see Sections 2 and 4).

2. The new terms of the system must follow the rule of "genus" and "specific difference":

Description of a definition by genus and specific difference: A definition of a mathematical notion must contain first a broader category or concept to which the defined notion belongs, the genus, then it needs to be distinguished from other items in that category by specific characteristics, the differentia. The genus must have been previously defined or is one of the undefined (primary) terms.

Example: A *triangle* is a polygon (genus) with three vertices (points-left undefined).

3. The system has a number of statements called axioms, which are considered to be true. The axioms relate to the terms and concepts of the system.

Example 1: *Euclid's Postulates*

(1) To draw a straight line from any point to any other
(2) To produce a finite straight line continuously in a straight line
(3) To describe a circle with any center and any distance
(4) That all right angles are equal to each other
(5) That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines if produced indefinitely, meet on that side on which are the angles less than the two right angles

Example 2: Hilbert's Axiomatic system discussed in the next section.

4. The system may contain a number of statements, called Theorems, which are derived logically from the axioms.

Example 1: In a right triangle whose leg's length are a and b and hypothenuse c, we have $a^2 + b^2 = c^2$. (Euclidean Geometry)

Example 2: The sum of the interior angles in a triangle is less than or equal $180°$ (Neutral Geometry).

1.1. Example of an axiomatic system.

Consider the following axiomatic system: *point*, *line* and the relation *on* are left undefined.

Axiom 1: There are exactly 12 lines in this geometry.

Axiom 2: Every line is on exactly two points and every two distinct points are on at most one line.

Axiom 3: Every point is on exactly three lines.

Theorem 1: There are exactly 8 points in this geometry.

Proof. Let us denote by V the number of points in this geometry. By Axiom 3, every point is on exactly three lines. So, if we multiply V by 3, by Axiom 2, we get the number of lines counted twice, one for each of the two points on it. Hence we have $3V = (12)(2)$. Solving for V, we get $V = 24/3 = 8$.

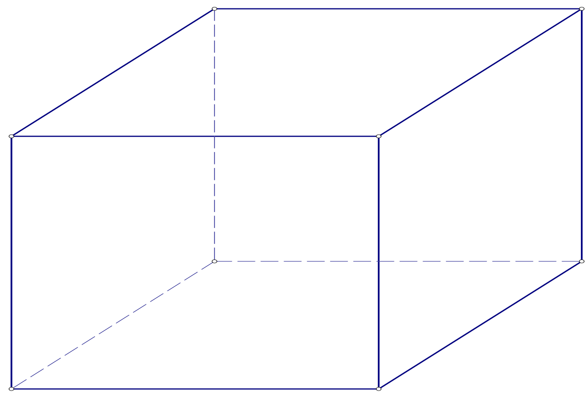

Figure 1

1.2. **Models.** A model of an axiomatic system is an interpretation of the undefined terms and of the relations between them, in such a way, that all axioms can be checked as being true statements. For the example we had earlier, we can take a cube for instance (see Figure 1), and consider its vertices as the "points" of the axiomatic system, the edges as the "lines" and the relation on to mean that two vertices are the endpoints of an edge. Then we have $4(3) = 12$ edges, and every vertex is on exactly three edges... This means the axioms are satisfied.

Every Theorem in an axiomatic system must be a true statement in every of its models. However, if a certain property takes place in a particular model, it does not mean it is a Theorem in the axiomatic system. For instance, we observe that in the cube model of our axiomatic system, one can find a path through edges between every two vertices. This concept is usually referred to the graph as being *connected*. One can see that this is actually not a Theorem, by constructing another model in which this property is not true. Let us consider the model of two tetrahedrons (see Figure 2) with the same interpretations for "points" and "lines" as before. In this second model two vertices that are on different tetrahedron cannot be connected by a path.

This brings us to another concept about models: we say two models are **isomorphic** if there exists a correspondence between the undefined terms so that the relationships between terms are preserved. The examples above are clearly not isomorphic since every property of one has to be valid in the other and that is not the case.

Models are called **concrete** if they are picked from the real world. If the model uses some other mathematical construction is called an *abstract model*. An axiomatic system is called **independent** if every of its axioms cannot be derived from the others. Using models one can show the independence of the axioms.

Our example above is independent. Indeed, first let us construct a model in which Axiom II and Axiom III are satisfied but Axiom I is not. For this purpose we may take a triangular pyramid as in Figure 2 for this.

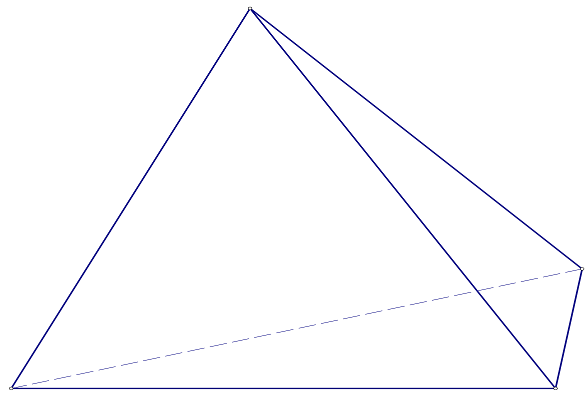

Figure 2

To show that Axiom II is independent, we need a model in which Axiom I and Axiom III are satisfied but Axiom II is not. We consider the model in Figure 3. There are three lines which are on four points.

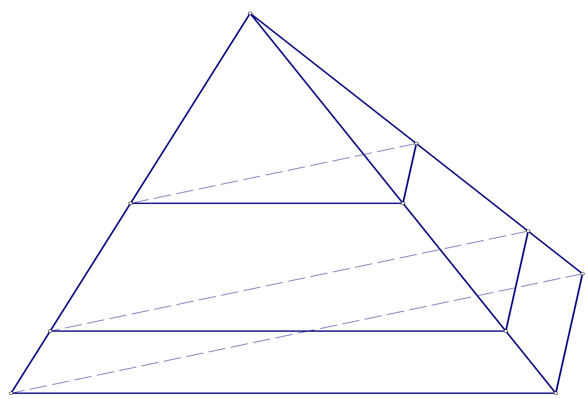

Figure 3

Finally for the independence of Axiom III, we just take a polygon with 12 vertices.

An axiomatic system is called **consistent** if one cannot derive a statement and its negation based on the axioms. The consistency is usually shown by constructing concrete models and if only abstract models exist, the system is called *relatively consistent*. So, our system is consistent and independent.

Another characteristic of an axiomatic system is to be **complete**, i.e. any statement or its negation can be derived within the system. In other words, any possible question about it can be answered within the system. An axiomatic system for which every two models are isomorphic is said to be **categorical**, and this property is known to be enough for the system to be complete. A system in which all the terms and relations are finite is complete.

Theorem 2: All possible models for our system (non-isomorphic) are those shown in Figure 4.

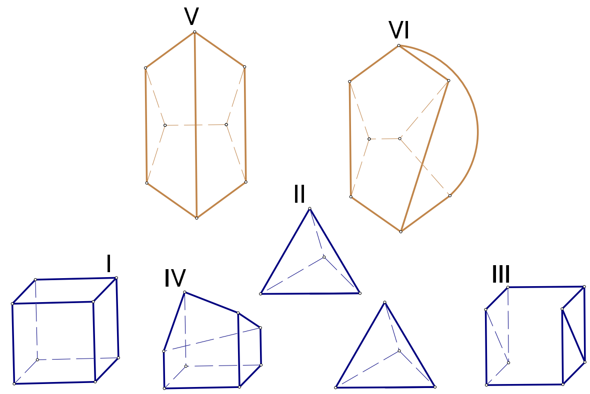

Figure 4

Proof. We take a point A and by Axiom 3, we may consider the three lines containing it: $\overline{AB}$, $\overline{AC}$, and $\overline{AD}$. By Axiom 2, these four points are distinct (A, B, C and D). Since we have 12 lines, having two more for each of the points B, C and D it leaves three that are not connected with these four points. These three lines have to share some points otherwise it will create six more points. We have then three different situations (Figure 5) : either there are three lines which are connected to a point, say E, and do not have any point in common with A, B, C and D (Case I in Figure 5); or, the three lines are forming a triangle say, EFG and the fourth point H is connected to each of the A, B, C and D (Case II in Figure 5); or finally the three lines are sharing only two of the four extra vertices (Case III in Figure 5).

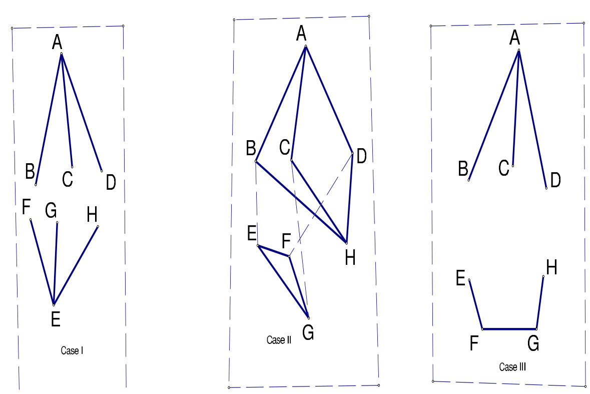

Figure 5

The Case II leads to Model VI. For the first case we have a problem similar to the one we have started. In how many ways can we connect 6 points (B, C, D, F, G and H) in such a way every point is on two segments and every segment is on exactly two points? We proceed as before and reduce the problem to two vertices, each with two edges and the four edges being disjoint (see Figure 6).

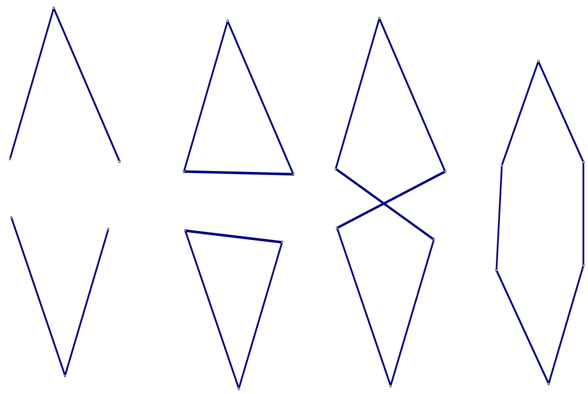

Figure 6

This leads to Model II (Case I(a), Figure 7) and Model III (Case (I(b)), if the two triangles are used in the two possible ways (either all three vertices of one triangle are B, C and D or only two).

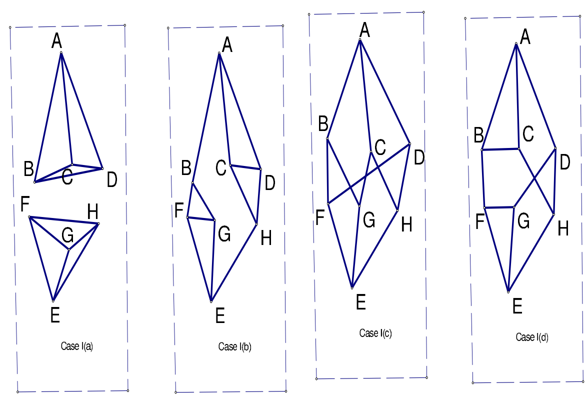

Figure 7

Finally, using an hexagon as in Figure 6 to connect the six vertices, either we can alternate the vertices B, C and D with the set of three vertices F, G and H to obtain Model I (Case I(c) in Figure 7); or, we can stay with on side or two sides in the vertices B, C and D in which case we obtain Model V (only two triangles, Case I(c) in Figure 7).

One can similarly analyze Case III. The various cases are depicted in Figures 8 and 9. ■

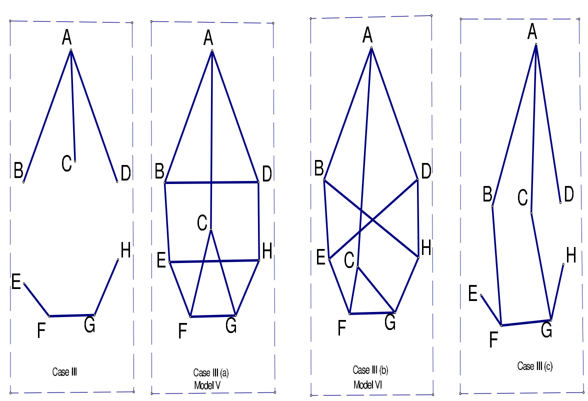

Figure 8

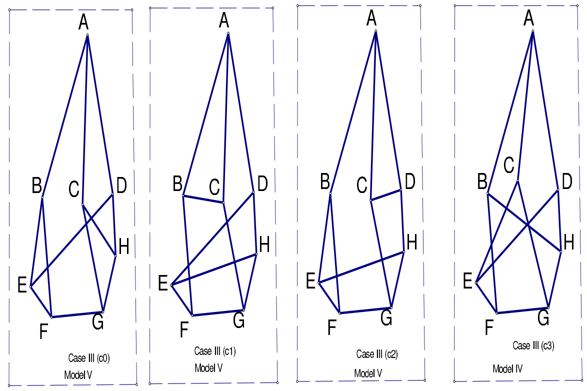

Figure 9

Problem 2: *Prove that if the system is connected then it contains a Hamiltonian path (a path that goes through all vertices exactly once).*

Problem 3: *How many isomorphisms of models can we find for each model ?.*

2. Hilbert's Axiomatic System for Euclidean Geometry

The following version of Hilbert's axioms is meant to be as independent as possible although we do not intend to prove their independence. We preferred a more wordy version of the axioms as opposed to using technical or/and formal notation.

Undefined terms : *point, line, plane, lie (incidence of point and line), between, congruence*

Axioms of Incidence (Connection)

- I-1. Every two distinct points determine a unique line that contains them.
- I-2. There exist at least two points on a given line. There exist at least three points that do not lie on a line.
- I-3. For any three points that do not lie on the same line, there exists a plane, that contains each of these three points. For every plane, there exists a point that it contains.

Axioms of order (betweeness)

A *line segment* determined by two points is defined as the set of all points on the line determined by those two points (Axioms I-1, I-2) that are between these points (called the endpoints of the line segment). We say that a line *cuts* a line segment if the line segment has a point on that line. A *triangle* is a union of three line segments determined by each two of three distinct non-collinear points. As for notation, the line segment determined by A and B is denoted by $\overline{AB}$. The points A and B are called endpoints of $\overline{AB}$.

- II-1. If one point is between two others (the order of these two being irrelevant) then all three points must be distinct and collinear.
- II-2. For every two distinct points A and B, then there is at least one point C on $\overleftrightarrow{AB}$ such that B is between A and C.

- II-3 If three distinct points are collinear then one and only one is between the other two.
- II-4 If a line cuts one side of a triangle then it must cut one of the other two sides or pass through a vertex.

A *ray* defined by two distinct points, A and B, and denoted $\overrightarrow{AB}$ is defined to be the set of all those points P on the line $\overleftrightarrow{AB}$ different of A and such that A is not between P and B. The point A is called *the endpoint* of the ray. As a result of the previous axioms one can talk about the two rays determined on a line by a point on it and about the two half planes determined by this line in any plane containing it. An *angle* is a union of two rays whose endpoints coincide.

Axioms of congruence

- III-1 Given a line, a point on the line and a segment, there exists two points on the line that determine with the given point two more segments congruent with the given one.
- III-2 The congruence of segments is a transitive relation.
- III-3 The congruence of segments is compatible with the juxtaposition of segments.
- III-4 Given a ray and an angle, then there exist two angles sharing the given ray congruent to the given angle.
- III-5 The hypothesis of SAS case of congruency of triangles insures that at one of the angles in one triangle is congruent to the corresponding angle in the second triangle.

Axiom of parallels

- IV-1 Given a line and a point not on that line, there exist at most one line parallel to the given line and containing the given point.

Axioms of continuity (completeness)

- V-1 (Archimedes Axiom) Given two segments one can find another segment containing the first and congruent to a number of copies of the second.
- V-2 The construction of every plane is maximal in the sense that there is no bigger plane in which all previous axioms are satisfied.

3. Some Theorems

Theorem 3.1. *A line cannot cut all three sides of a triangle unless it contains a vertex of the triangle.*

Proof. By way of contradiction let us assume that there exists a line ℓ and a triangle ABC such that A, B and C are not on ℓ and $\ell \cap \overline{AB} = D$, $\ell \cap \overline{BC} = E$, and $\ell \cap \overline{AC} = F$. By Axiom I-1, two lines if they are distinct they can intersect at most at one point. So, the points D, E

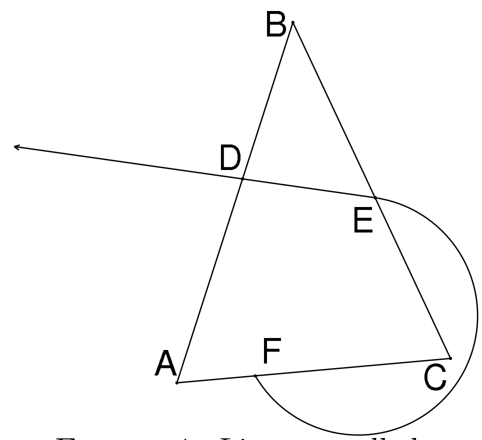

FIGURE 1. Line cuts all three sides

and F must be distinct, otherwise ℓ must contain one of the vertices of $\triangle ABC$. By Axiom II-3, one and only one of the points D, E and F is between the other two. Without loss of generality we may assume that E is between D and F as in the Figure 1. Then we apply Axiom II-4 to $\triangle ADF$ and line $m := \overleftrightarrow{BC}$. Observe that since m cuts one of the sides of $\triangle ADF$ (it cuts $\overline{DF}$ at E), it must cut one of the other two or contain a vertex of $\triangle ADF$. Either one of these possibility is basically excluded by our hypothesis on the configuration. The contradiction obtained shows that the statement must be true. ■

Theorem 3.2. *Suppose O is on $\overline{AB}$. Then*

(a) if C is on $\overline{AO}$ implies A is not between C and B;

(b) if C is on $\overline{AB}$ implies A is not between O and C.

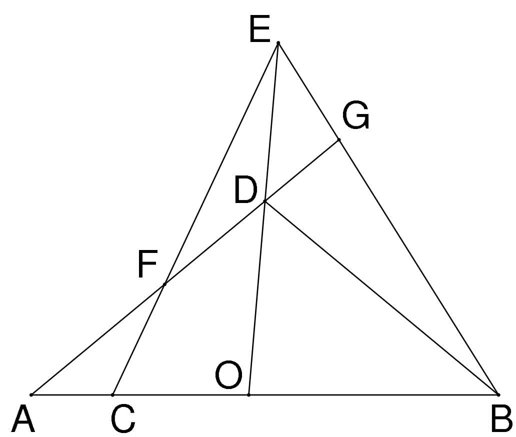

FIGURE 2. O-D-E construction

Proof. (a) Assume by contrary that A is between C and B. By Axiom I-2 there exist a point D not on the line determined by A and B and by Axiom II-2, there exists a point E such that D is between O and E (Figure 2). Let us denote the line $\overleftrightarrow{AD}$ by m. By Axiom II-4, applied to line m and triangle $\triangle ECO$, m cuts $\overline{EC}$ say at F. Also, Axiom II-4 applied to m and triangle $\triangle EOB$ implies that m cuts $\overline{EB}$ at a point say G. Now, we obtained a contradiction with Theorem 3.1 since m cuts all sides of the triangle $\triangle ECB$.

(b) We use the whole construction as above and in the end we use line m and triangle $\triangle ECO$ instead. ■

Theorem 3.3. *If O is on $\overline{AB}$, then*

$$\overline{OA} \cap \overline{OB} = \emptyset.$$

Proof. Assume by contrary that the intersection is not empty and let C be on both segments. Since $C \in \overline{OA}$, by Theorem 3.2, A is not between B and C. Similarly since $C \in \overline{OB}$, by Theorem 3.2, B is not between A and C. By Axiom II-3, it must be true that $C \in \overline{AB}$. But then we get a contradiction with Theorem 3.1, since line $\overleftrightarrow{EC}$ cuts all sides of the triangle $\triangle DAB$. ■

Now we can show a fact similar to Theorem 3.2.

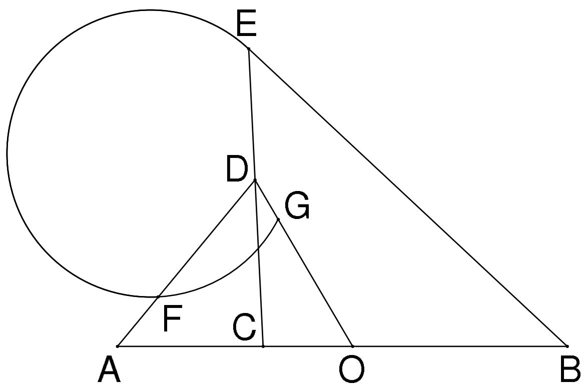

FIGURE 3. C-D-E construction

Theorem 3.4. *Suppose O is on $\overline{AB}$ and C is on $\overline{AO}$. Then*
(a) B is not between A and C;
(b) B is not between O and C.

Proof. (a) Assume by contrary that B is between A and C. By Axiom I-2 there exist a point D not on the line determined by A and B and by Axiom II-2, there exists a point E such that D is between C and E (Figure 3). Let us denote the line $\overleftrightarrow{BE}$ by m. By Axiom II-4, applied to line m and triangle $\triangle DAC$, m must cut $\overline{AD}$ say at F. Also, Axiom II-4 applied to m and triangle $\triangle DAO$ implies that m cuts $\overline{DO}$ at a point say G. Finally, Axiom II-4 applied to m and triangle $\triangle DCO$ gives that m cuts $\overline{CO}$. Now, we obtained a contradiction with Theorem 3.3 since B is on opposite segments: $\overline{AC}$ and $\overline{CO}$.

For part (b) the argument goes somewhat similarly as above with the exception we show first show the existence of G and then F. In the end we end up with the same contradiction. ■

Putting together Theorem 3.2, Theorem 3.3 and Theorem 3.4 we obtain as a corollary the first order relation of four points on a line.

Corollary 3.1. [Order relation of four collinear points (part 1)] *O is on $\overline{AB}$ and C is on $\overline{AO}$. Then*
(a) C is between A and B;
(b) O is between C and B.

Proof. Part (a) follows directly from Theorem 3.2 and Theorem 3.4. For the second part, let us observe that $B \in \overline{OC}$ is excluded by Theorem 3.4 and $C \in \overline{OB}$ is ruled out by Theorem 3.3. ■

We are going to use notation $A - B - C$ for B in between A and C.

Theorem 3.5. **[Order relation of four collinear points (part 2)]** *If $A - B - C$ and $B - C - D$ then the four points are distinct and collinear. In addition, we must have $A - C - D$ and $A - B - D$.*

Proof. It is easy to show that all points are collinear and distinct. It is clear that it is enough to show one of the relationships. Say we want to show $A - C - D$. By Axiom II-3, we need to exclude $A - D - C$ and $D - A - C$. If we have $D - A - C$ then by Corollary 3.1 we must have $D - B - C$ which contradicts the hypothesis. If we have $A - D - C$ then we cannot have $B - C - D$ by Theorem 3.2 (part b). ■

At this point we can go one step further and show the separation of the line theorem.

Theorem 3.6. **[Separation of line]** *If O is on $\overline{AB}$, then*

$$\overleftrightarrow{AB} = \overrightarrow{OA} \cup \{O\} \cup \overrightarrow{OB} \quad \text{and} \quad \overrightarrow{OA} \cap \overrightarrow{OB} = \emptyset. \tag{1}$$

Proof. Exercise

So far, it seems like a lot of work and we accomplished very little.

If two points A and B are not on a line ℓ then we say that they are *on the same side of* ℓ if $\ell \cap \overline{AB} = \emptyset$ or *on opposite sides of* ℓ if $\ell \cap \overline{AB} \neq \emptyset$.

Theorem 3.7. *Let A, B and C three points not on the line ℓ. If*
(a) A and B are on the same side of ℓ, B and C are also on the same side of ℓ,
or
(b) A and B are on opposite sides of ℓ, B and C are also on opposite sides of ℓ,
then A and C are on the same side of ℓ.

Proof. Exercise

4. Birkhoff's Axioms for Euclidean Geometry

References

[1] E.C. Wallace and S. F. West, *Roads to Geometry*, Printice Hall, Second Edition, 1998