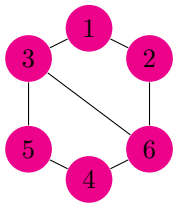
Class notes Math 5125

January 27th, 2024

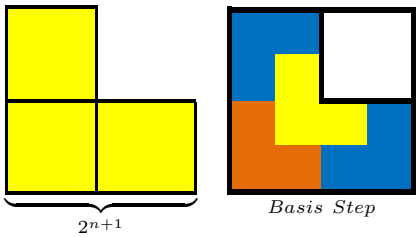
1 Induction



"Yesterday,
all my troubles seemed so far away
Now it looks as though they're here to stay
Oh, I believe in yesterday."

Some standard and some less standard examples of problems done by induction.

Problem 1. *For every $n \in \mathbb{N}$ the chess board of dimensions $2^{n+1} \times 2^{n+1}$ without a big square corner of dimensions $2^n \times 2^n$ can be tiled with L-shape tiles.* Prove this by induction.



Shape of the board

Solution: For the Basis Step, we take a board which is 4×4 without the right-up corner 2×2 as in the picture above. We show in the picture how this can be covered with L-shape tiles. For the Induction Step let us assume we can cover a board which is $2^{n+1} \times 2^{n+1}$ without a big corner square of dimension $2^n \times 2^n$, for some $n \geq 1$. Let us take a board $2^{n+2} \times 2^{n+2}$ without a big corner square of dimension $2^{n+1} \times 2^{n+1}$. We can subdivide this board in 4 smaller boards $2^{n+1} \times 2^{n+1}$ without a big corner square of dimension $2^n \times 2^n$, as in the Basis Step. Each one of these can be tiled by the induction hypothesis. An example is shown in a 8×8 board (Figure 2). Hence by the PMI the statement is true for every $n \in \mathbb{N}$. ■

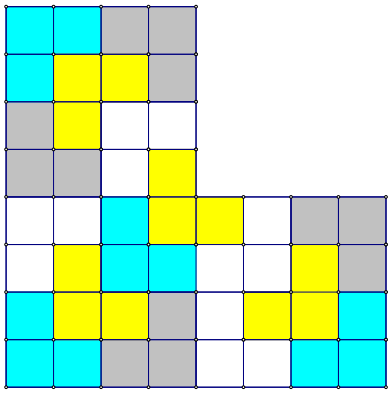


Figure 2

Problem 2. *Use induction to prove that for every $n \in \mathbb{N}$, we have*

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}. \tag{1}$$

Solution: Basis Step: For $n = 1$, (1) is equivalent to

$$1^2 = \frac{1(4-1)}{3} \Leftrightarrow 1 = 1,$$

which shows that (1) is correct in this case.
Induction Step: Assume that (22) is true for some $n \geq 1$. Then

$$1^2 + 3^2 + ... + (2n-1)^2 + (2n+1)^2 = \frac{n(4n^2-1)}{3} + (2n+1)^2,$$

but

$$\frac{n(4n^2-1)}{3} + (2n+1)^2 = \frac{n(2n-1)(2n+1) + 3(2n+1)^2}{3} = \frac{(2n+1)(2n^2-n+6n+3)}{3}$$

$$= \frac{(2n+1)(2n^2+5n+3)}{3} = \frac{(2n+1)(n+1)(2n+3)}{3} = \frac{(n+1)[(2n+2-1)(2n+2+1)]}{3}.$$

Therefore,

$$1^2 + 3^2 + ... + (2n-1)^2 + (2n+1)^2 = \frac{(n+1)[4(n+1)^2-1]}{3},$$

which is exactly (1) for $n+1$ instead of n.
So, by the PMI we have (22) true for every $n \in \mathbb{N}$. ■

Problem 3. *Prove by induction that*

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + ... + \frac{1}{2n},\ n \geq 1. \tag{2}$$

Solution: Basis Step: If $n = 1$, $LHS(21) = \frac{1}{1} - \frac{1}{2} = \frac{1}{2}$ and $RHS(21) = \frac{1}{2}$, which means (21) is true for $n = 1$.
Induction Step: Let us assume that (2) is true for some $k \in \mathbb{N}$. Hence, we have

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{k+1} + ... + \frac{1}{2k} \tag{3}$$

We add the term missing to get the appropriate sum in (2) (for $n = k+1$):

$$LHS(n = k+1) = \tfrac{1}{1} - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + ... + \tfrac{1}{2k-1} - \tfrac{1}{2k} + \tfrac{1}{2k+1} - \tfrac{1}{2k+2} =$$

$$\tfrac{1}{k+1} + ... + \tfrac{1}{2k} + \tfrac{1}{2k+1} - \tfrac{1}{2k+2} =$$

$$\tfrac{1}{k+2} + ... + \tfrac{1}{2k} + \tfrac{1}{2k+1} + (\tfrac{1}{k+1} - \tfrac{1}{2k+2}).$$

But the last two terms $\frac{1}{k+1} - \frac{1}{2k+2}$ can be simplified to $\frac{1}{2k+2}$ and so $LHS(n = k+1) = RHS(n = k+1)$. Hence, we must have (21) for $n = k+1$. By the PMI we know that (2) must be true for every $n \geq 1$. ■

Problem 4 **:** Prove by induction that, there exists a sequence of 1's or 2's, a_1, a_2, ..., such that the number $a_n a_{n-1} ... a_2 a_1$ is divisible by 2^n.

Solution: Let us denote the number $a_n a_{n-1}...a_2 a_1$ by G_n. We need to show the existence of $\{a_n\}$ of digits 1's or 2's such that

$$G_n \text{ is divisible by } \quad 2^n, \ n \in \mathbb{N}. \tag{4}$$

We prove (4) using induction on n.

Basis Step: If $n = 1$ then we can take $a_1 = 2$ and so (4) is satisfied ($G_1 = a_1 = 2$ is divisible by $2^1 = 2$). If $n = 2$, then we can take $a_2 = 1$ and a_1 as before. Then (4) is satisfied since $G_2 = a_2 a_1 = 12$ is divisible by $2^2 = 4$.

Inductive Step: Let us assume the $a_1 = 2$, $a_2 = 1$, ..., a_n have been found, so that (4) is correct, for some $n \geq 2$. Then, we can write $G_n = 2^n q$ for some $q \in \mathbb{N}$. This is our induction hypothesis. Then we need to show that $a_{n+1} \in \{1, 2\}$ exists such that

$$G_{n+1} \text{ is divisible by } \quad 2^{n+1}. \tag{5}$$

We observe that $10^n = 2^n \cdot 5^n$ and so

$$G_{n+1} = a_{n+1}10^n + G_n = a_{n+1}2^n \cdot 5^n + 2^n q = 2^n(a_{n+1}5^n + q).$$

Case I (q is and odd number): If q is odd then a_{n+1} must be equal to 1, becasue $5^n + q$ is then an even number. This implies

$$G_{n+1} = 2^n(a_{n+1}5^n + q) = 2^n(5^n + q) = 2^n(2s) = 2^{n+1}s$$

and so (5) is satisfied.

Case II (q is even): If q is even then a_{n+1} must be equal to 2, becasue $2 \cdot 5^n + q$ is then an even number. This implies

$$G_{n+1} = 2^n(a_{n+1}5^n + q) = 2^n(2 \cdot 5^n + q) = 2^n(2s) = 2^{n+1}s$$

and so (5) is satisfied. This ends the inductive step.

By the PMI, (4) is true for every n. □

This sequence is in the Online Encyclopedia of Integer Sequences as A023396 or A053312

Problem 5**:** Use a combinatorial proof to show that

$$\sum_{k=q}^{n} \binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}, \quad n \geq q. \tag{6}$$

Solution: Let us consider the set A of all possible ways of making a committee (order is not important) of q permanent members chosen from n people and in addition we may have as many at-large members. Two committess are considered different if the permanent members are not the same or if the at-large members are not the same (the totality of the committee may be the same but that, by itself, does not make the two to be the same). We can build such a committee in two different ways.

We can pick the permanent members first, and that can be done in $\binom{n}{q}$ ways. Then we add any numbers of the remaining $n - q$ people to the committee. The number of subsets of a set with m elements is 2^m, so we can form $\binom{n}{q}2^{n-q}$ such committees. That is the right hand side of (6).

We can also, pick a committee of $k \geq q$ people and then choose the q permanent ones from that. We have $\binom{n}{k}$ to pick the whole committee and then for each such choice we have $\binom{k}{q}$ ways to pick the q permanent members. This gives a total of $\binom{n}{k}\binom{k}{q}$. This is the number for each $k \geq q$. Hence, adding them together gives $\sum_{k=q}^{n}\binom{n}{k}\binom{k}{q}$ which is the left-hand side of (6). □

Remark: Let us observe that the idea of proof is equivalent to counting in two different ways the number of pairs (U, V), $U \subset V \subset \{1, 2, ..., n\}$ with $|U| = q$ (number of elements is q). Try to use this technique to

arrive at an identity counting all triples (U, V, W), $U \subset V \subset W \subset \{1, 2, ..., n\}$, with $|U| = q$ and $|V \setminus U| = r$ $(q + r \leq n)$.

$$\sum_{k=q+r}^{n} \binom{n}{k}\binom{k}{q}\binom{k-q}{r} = 2^{n-q-r}\binom{n}{q}\binom{n-q}{r}, \quad n \geq q,\ r \geq 0$$

Algebraic proof: Using the formula for the combinations we get

$$\sum_{k=q}^{n} \binom{n}{k}\binom{k}{q} = \sum_{k=q}^{n} \frac{n!}{k!(n-k)!}\frac{k!}{q!(k-q)!} = \sum_{k=q}^{n} \frac{n!}{(n-k)!q!(k-q)!} = \frac{n!}{q!(n-q)!}\sum_{k=q}^{n} \frac{(n-q)!}{(n-k)!(k-q)!} \implies$$

$$\sum_{k=q}^{n} \binom{n}{k}\binom{k}{q} = \binom{n}{q}\sum_{k=q}^{n} \binom{n-q}{k-q} = \binom{n}{q}\sum_{i=0}^{n-q} \binom{n-q}{i} = 2^{n-q}\binom{n}{q}.$$

More practice problems for induction

The Fibonacci sequence denote by $\{F_n\}_{n\geq 0}$ is defined recursively by $F_0 = 0$, $F_1 = 1$ and for every $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

1. Binet's formula:
$$F_n = (\alpha^n - \beta^n)/\sqrt{5}, \text{where } \alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}. \tag{7}$$
2. $F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-1}{2} + ...$
3. For all $n \in \mathbb{N}$, we have $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$.
4. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$, for all $n \in \mathbb{N}$.
5. For all $n \in \mathbb{N}$, we have
$$\sum_{k=0}^{n} \frac{1}{2^k}\binom{n+k}{k} = 2^n.$$
6. For $m, n \in \mathbb{N}$ we get
$$F_{m+n} = F_m F_{n-1} + F_n F_{m+1}.$$
7. Given a set A of $2n$ $(n \geq 2)$ points in the plane and $n^2 + 1$ segments with endpoints in A, show that there exist three points in A which form a triangle with some of these segments. In the language of graphs, every simple graph on $2n$ vertices and $n^2 + 1$ edges contains a triangle.
8. For every $n \in \mathbb{N}$, $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.
9. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} < 2 - \frac{1}{n}$, if $n \geq 2$.
10. $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, $n \geq 1$.

2 Combinatorial Objects

1. `Permutations :` Notation P_n

Definition 1: A permutation of a set A is a bijection f from A to A, i.e., f is one-to-one and onto.

A function is one-to-one if for different inputs it gives different outputs, and it is onto if the range is its codomain. We observe that what is characteristic about permutations is that all objects are used only one time in any order possible. Sometime we call this the set of all bijections of a set of n objects. On a finite set a bijection is equivalent with a one-to-one (injection) or an onto (surjection) function. So, this is also the set of all surjections on a set of n objects.

Example: The permutations of the objects $\{a, b, c, d\}$ are

$$\begin{bmatrix} \{a,b,c,d\} & \{a,b,d,c\} & \{a,c,b,d\} & \{a,c,d,b\} & \{a,d,b,c\} & \{a,d,c,b\} \\ \{a,d,b,c\} & \{a,d,c,b\} & \{b,a,c,d\} & \{b,a,d,c\} & \{b,c,a,d\} & \{b,c,d,a\} \\ \{b,c,a,d\} & \{b,c,d,a\} & \{b,d,a,c\} & \{b,d,c,a\} & \{c,a,b,d\} & \{c,a,d,b\} \\ \{c,a,b,d\} & \{c,a,d,b\} & \{c,b,a,d\} & \{c,b,d,a\} & \{c,d,a,b\} & \{c,d,b,a\} \end{bmatrix}$$

From an algorithmic point of view, there is this question of "how do we generate all these permutations?". The proof of the next theorem actually gives us a way to do that but the table above, constructed with Mathematica has a different rule of generating them.

Theorem 1: `The number of permutations of` n `objects is` $n!$`, i.e.,`

$$P_n = n!. \tag{8}$$

Proof: We proceed by induction on n.
Basis Step: If $n = 1$, we have only one object and there is only one way to permute so $P_1 = 1$ and $1! = 1$ so (21) is correct.
Inductive Step: We begin with the induction hypothesis which is, we assume (21) is correct for some $n \geq 1$. We need to show that

$$P_{n+1} = (n+1)!. \tag{9}$$

So we know that there are $n!$ permutations of n objects. We then take $n+1$ objects, say $\{1, 2, 3, ..., n, n+1\}$. Suppose we take one of the $n!$ permutations of the first n objects say $a_1 a_2 \cdots a_n$. Then we place $n+1$ in all possible positions

$$(n+1)a_1 a_2 \cdots a_n, \quad a_1(n+1)a_2 \cdots a_n, \quad \cdots, \quad a_1 a_2 \cdots a_n(n+1),$$

which gives $n+1$ new permutations of the $n+1$ objects. Let us observe that these are all distinct and distinct for each two permutations $a_1 a_2 \cdots a_n$ and $b_1 b_2 \cdots n_n$. The reason is the folowing: they either have (n+1) on different positions or if these positions coincide then they must be different on at least one position since $a_1 a_2 \cdots a_n$ and $b_1 b_2 \cdots n_n$ are distinct. Therefore, we have at least $n!(n+1) = (n+1)!$ permutations of $n+1$ objects. We cannot have more than that becasue for every permutation of $n+1$ objects we can see that it can be constructed in this way. Therefore, the number of permutations of $n+1$ objects is given by (22). This ends the inductive step.

By the PMI, (21) is true for every n. □

2. `Arrangments or k-permutations:` Notation A_n^k

Definition 2: A k-permutation of a set A is one-to-one function f from $\{1, 2, ..., k\}$ to A.

If A is finite, say $A = \{a_1, a_2, ..., a_n\}$, a k-permutation is just an ordered list of distinct k elements of A: $(a_{i_1}, a_{i_2}, ..., a_{i_k})$.

Example: The next matrix shows all arrangements of two objects selected from the set $\{a, b, c, d, e, f\}$:

$$\begin{bmatrix} \{a,b\} & \{a,c\} & \{a,d\} & \{a,e\} & \{a,f\} & \{b,a\} \\ \{b,c\} & \{b,d\} & \{b,e\} & \{b,f\} & \{c,a\} & \{c,b\} \\ \{c,d\} & \{c,e\} & \{c,f\} & \{d,a\} & \{d,b\} & \{d,c\} \\ \{d,e\} & \{d,f\} & \{e,a\} & \{e,b\} & \{e,c\} & \{e,d\} \\ \{e,f\} & \{f,a\} & \{f,b\} & \{f,c\} & \{f,d\} & \{f,e\} \end{bmatrix}$$

Theorem 2: `The number of arrangements of` k `objects chosen from` n `objects is equal` $n(n-1)(n-2)\cdot(n-k+1)$`, i.e.,`

$$A_n^k = \frac{n!}{(n-k)!}, \quad n \geq k.$$

In the example above, we have $A_n^k = A_6^2 = 6(5) = 30$.

3. `Combinations:` Notation $\binom{n}{k}$

Definition 3: A k-subset of a set A is a subset which has k elements. The total number of k-subsets of a set A having n elements, or we say, of cardinality n, is denoted by $\binom{n}{k}$.

In this case, the order is irrelevant, as we wellknow about the elements of a set.

Example: The next matrix shows all ways to choose 3 objects from $\{a, b, c, d, e, f\}$:

$$\begin{bmatrix} \{a,b,c\} & \{a,b,d\} & \{a,b,e\} & \{a,b,f\} & \{a,c,d\} \\ \{a,c,e\} & \{a,c,f\} & \{a,d,e\} & \{a,d,f\} & \{a,e,f\} \\ \{b,c,d\} & \{b,c,e\} & \{b,c,f\} & \{b,d,e\} & \{b,d,f\} \\ \{b,e,f\} & \{c,d,e\} & \{c,d,f\} & \{c,e,f\} & \{d,e,f\} \end{bmatrix}$$

We have $\binom{6}{3} = \frac{6!}{3!(3!)} = \frac{6(5)(4)}{2(3)} = 20$ 3-subsets of a set with 6 objects.

Theorem 3: `The number of combinations of` k `objects chosen from` n `objects is equal`

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{10}$$

There is a symmetry that can be derived right away from (10):

$$\binom{n}{k} = \binom{n}{n-k}.$$

Computational example:

$$\binom{8}{4} = \frac{8!}{4!(4!)} = \frac{8(7)(6)5}{2(3)4} = 70.$$

One of the most famous formula in combinatorics that involves these numbers is the Binomial Formula:

$$\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots \binom{n}{n}b^n = (a+b)^n, \quad a, b \in \mathbb{R}, n \in \mathbb{N}. \tag{11}$$

A good exercise here is to prove this by induction on n. This proof is based on the Pascal Recurrence identity:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \quad n \geq k-1, \ k \geq 1. \tag{12}$$

which is at the heart of building the Pascal's Triangle :

$$
\begin{array}{ccccccccccccccccccc}
 & & & & & & & 1 & & & & & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & & & & & \\
 & & & & & 1 & & \boxed{2} & & 1 & & & & & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & & & & & \\
 & & & 1 & & 4 & & \boxed{6} & & 4 & & 1 & & & & & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & & & & \\
 & 1 & & 6 & & 15 & & \boxed{20} & & 15 & & 6 & & 1 & & & & & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & & & & \\
 & 1 & & 8 & & 28 & & 56 & & \boxed{70} & & 56 & & 28 & & 8 & & 1 & \\
1 & & 9 & & 36 & & 84 & & 126 & & 126 & & 84 & & 36 & & 9 & & 1 \\
 & 1 & & 10 & & 45 & & 120 & & 210 & & \boxed{252} & & 210 & & 120 & & 45 & & 10 & & 1 \\
1 & & 11 & & 55 & & 165 & & 330 & & 462 & & 462 & & 330 & & 165 & & 55 & & 11 & & 1 \\
 & 1 & & 12 & & 66 & & 220 & & 495 & & 792 & & \boxed{924} & & 792 & & 495 & & 220 & & 66 & & 12 & & 1 \\
1 & & 13 & & 78 & & 286 & & 715 & & 1287 & & 1716 & & 1716 & & 1287 & & 715 & & 286 & & 78 & & 13 & & 1 \\
 & 1 & & 14 & & 91 & & 364 & & 1001 & & 2002 & & 3003 & & \boxed{3432} & & 3003 & & 2002 & & 1001 & & 364 & & 91 & & 14 & & 1 \\
1 & & 15 & & 105 & & 455 & & 1365 & & 3003 & & 5005 & & 6435 & & 6435 & & 5005 & & 3003 & & 1365 & & 455 & & 105 & & 15 & & 1
\end{array}
$$

Another way of writing the Binomial formula (11) is with the help of summation notation:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \quad a, b \in \mathbb{R}, n \in \mathbb{N}. \tag{13}$$

As a result of this formula, the combination numbers are also refered to as the binomial coefficients.

4. `Tuples or permutations with repetitions allowed` We will refer to these objects mostly as all words (or strings) of length k from the alphabet A.

Definition 4: A k-tuple with entries from a set A is an arbitrary function f from $\{1, 2, 3, ..., k\}$ with values in A. The set of all fuctions is usually denoted by A^k.

Example: Here is the set $\{A, B\}^4$:

$$
\begin{bmatrix}
\{A,A,A,A\} & \{A,A,A,B\} & \{A,A,B,A\} & \{A,A,B,B\} \\
\{A,B,A,A\} & \{A,B,A,B\} & \boxed{\{A,B,B,A\}} & \{A,B,B,B\} \\
\{B,A,A,A\} & \{B,A,A,B\} & \{B,A,B,A\} & \{B,A,B,B\} \\
\{B,B,A,A\} & \{B,B,A,B\} & \{B,B,B,A\} & \{B,B,B,B\}
\end{bmatrix}
$$

Theorem 4: `The number of` n`-tuples that can be formed with` m`- element set` A `is equal to` m^n`, i.e.,`

$$|A^n| = m^n.$$

Corollary 1: The number of all subsets of a set A with n objects is equal to 2^n, i.e.,

$$|\mathcal{P}(A)| = 2^n.$$

Here we used the classical notation $\mathcal{P}(A)$ for the class of all subsets of a set A.

Proof: Let us take $A = \{1, 2, \cdots, n\}$. We think of a subset as a string of 0's or 1's. If lets say the subset S contains the element i we take a 1 on the ith position, and 0 otherwise. For example, if $A = \{1, 2, 3\}$ and $S = \{1, 3\}$ then the string associated is 101. This correspodence goes the other way too. If we take the string 011 this corresponds to the subset $\{2, 3\}$. So, there is a 1-1 correspondence between the set of subsets and the set of 0's or 1's strings. But this set the set of all n-tuples of the set $\{0, 1\}$ which has cardinality 2^n. □

Corollary 2: We have the identity

$$\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n.$$

Proof: The set of subsets can be partitioned into subsest of only k-elements which has cardinality $\binom{n}{k}$ and so, when we add all these numbers together we obtain, by Corolary 1, 2^n. □

5. `Combinations with repetition allowed:` Notation $\left(\binom{n}{k}\right)$.

Definition 5: A k-multisubset of a set A is a k-tuple with entries from A in which the order is irrelevant. The number of all such k-multisubsets of a set A with n-objects is denoted by $\left(\binom{n}{k}\right)$.

Example: In the next table are all combinations with repetition of choosing 3-multisubsets of the set $\{1,2,3,4,5\}$:

$$\begin{bmatrix} \{1,1,1\} & \{1,1,2\} & \{1,1,3\} & \{1,1,4\} & \{1,1,5\} & \{1,2,2\} & \{1,2,3\} \\ \{1,2,4\} & \{1,2,5\} & \{1,3,3\} & \{1,3,4\} & \{1,3,5\} & \{1,4,4\} & \{1,4,5\} \\ \{1,5,5\} & \{2,2,2\} & \{2,2,3\} & \{2,2,4\} & \{2,2,5\} & \{2,3,3\} & \{2,3,4\} \\ \{2,3,5\} & \{2,4,4\} & \{2,4,5\} & \{2,5,5\} & \{3,3,3\} & \{3,3,4\} & \{3,3,5\} \\ \{3,4,4\} & \{3,4,5\} & \{3,5,5\} & \{4,4,4\} & \{4,4,5\} & \{4,5,5\} & \{5,5,5\} \end{bmatrix}$$

Theorem 5: `The number of` k-`multisubsets of a set with` n `objects is given by the formula`

$$\left(\binom{n}{k}\right)=\binom{n+k-1}{k} \tag{14}$$

The proof of this theorem is based on the Stars and Bars method. We are going to deduce this from the following general theorem.

Theorem 6: `The number of` k`- ordered tuples of non-negative integers` $(x_1,x_2,...,x_k)$ `such that`

$$x_1+x_2+\cdots+x_k=n$$

`is given by`

$$\binom{n+k-1}{k-1} \tag{15}$$

Proof: The method of Stars and Bars goes in the following way. Let us think of a k- ordered tuple of non-negative integers $(x_1,x_2,...,x_k)$ satisfying $x_1+x_2+\cdots+x_k=n$ as a sequence of Stars and Bars:

$$\underbrace{***}_{x_1}|\underbrace{****}_{x_2}|\underbrace{*****}_{x_3}|\cdots\underbrace{**}_{x_k}.$$

Since we need to have k spaces with stars (their number in each slot is given by x_i), we only need $k-1$ bars as delimiters. Hence, the number of stars (equal to n since $x_1+x_2+\cdots+x_k=n$), plus the number of bars equals $n+k-1$. So, we see that this representation can be obtained from choosing the $k-1$ bars from $n+k-1$ places. We know that the number of such choices is equal to $\binom{n+k-1}{k-1}$. ■

Proof of Theorem 5: For each k-multisubset S of $\{1,2,3,...,n\}$ we associate a n-ordered tuple of non-negative integres $(x_1,x_2,...,x_n)$ such that $x_1+x_2+...+x_n=k$ in the following way: if i is in the k-multisubset S and appears j times, we define $x_i=j$. Since we only have k elements in S, we need to have $x_1+x_2+...+x_n=k$. This is exactly the setup of Theorem 6. Then the number of such tuples is $\binom{k+n-1}{n-1}$. This last combinatorial number is nothing else but $\binom{n+k-1}{k}$ because the binomial coefficients are symmetric. ■

Example: Problem 15, page 432, Rosen-textbook *How many solutions are there to the equation*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

where x_i, $i = 1, 2, 3, 4, 5$ *is a nonnegative integer such that*
(a) $x_1 \geq 1$*?*
(b) $x_i \geq 2$ *for all* $i = 1, 2, 3, 4, 5$*?*
(c) $0 \leq x_1 \leq 10$ *?*
(d) $0 \leq x_1 \leq 3$, $1 \leq x_2 \leq 4$ *and* $x_3 \geq 15$ *?*

Homework: Problem 16, page 432, Rosen-textbook *How many solutions are there to the equation*

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$$

where x_i, $i = 1, 2, 3, 4, 5, 6$ *is a nonnegative integer such that*
(a) $x_i \geq 1$ *for all* $i = 1, 2, 3, 4, 5, 6$*?*
(b) $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, $x_5 > 5$ *and* $x_6 \geq 6$ *?*
(c) $x_1 \leq 5$ *?*
(d) $x_1 < 8$ *and* $x_2 > 8$ *?*

6. `Multinomial coefficients:` Notation $\binom{n}{k_1, k_2, \ldots, k_r}$.

Definition 6: An r-partition of a set A is a sequence of r subsets of A, A_1, A_2, ..., A_r such that
(i) $A = A_1 \cup A_2 \cup \ldots \cup A_r$
(ii) if $i \neq j$, we have $A_i \cap A_j = \emptyset$
The number of all such k-partitions of a set A with n-objects into distinguished sets, A_i each having cardinatily $|A_i| = k_i$ is denoted by $\binom{n}{k_1, k_2, \ldots, k_r}$ and it is given by the following theorem.

Theorem 7: `The number of all` k`-partitions of a set` A `with` n`-objects into distinguished sets, with` $|A_i| = k_i$`, is given by`

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1! k_2! \cdots k_r!}. \tag{16}$$

The proof of this is based on the interpretation of $\binom{n}{k}$. An equivalent theorem is the following.

Theorem 8: `The number of different permutations of` n `objects, where there are` n_1 `indistinguishable objects of type 1,` n_2 `indistinguishable objects of type 2, . . . , and` n_k `indistinguishable objects of type` k`, is`

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}. \tag{17}$$

Example: How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

We can think of the 4 players as the types in the last theorem and the house is the last type. Hence the answer is

$$\binom{53}{5, 5, 5, 5, 32} = 1,478,262,843,475,644,020,034,240$$

a figure that is astronomical (diameter of the observable universe is estimated to about 8.8×10^{26} m).

Theorem 9: `The binomial formula can be generalized in the following way`

$$(a_1 + a_2 + \cdots + a_k)^2 = \sum_{l_1+l_2+\cdots+l_k=n} \binom{n}{l_1, l_2, ..., l_k} a_1^{l_1} a_2^{l_2} \cdots a_k^{l_k}. \tag{18}$$

For example the coefficient of $a^2b^3c^4$ in the expansion of $(a+b+c)^9$ is given by

$$\binom{9}{2,3,4} = \frac{9!}{2!3!4!} = \frac{5(6)(7)(8)9}{2(6)} = 5(4)(7)9 = 63(20) = 1260.$$

Excercise: Find the coefficient of $a^2b^3c^4$ in the expansion of $(3a+2b+c)^9$

7. `Stirling numbers of second type:` Notation $\left\{ {n \atop k} \right\}$.

Definition 7: The number of k-partitions of a set A with n distingushable elements, in which the partition sets are non-empty and the order is irrelevant is called the Stirling number of second kind and it is denoted by $\left\{ {n \atop k} \right\}$.

For example $\left\{ {5 \atop 3} \right\} = 25$ since, lets say, we have a set with 5 elements in it, $A = \{1,2,3,4,5\}$. We can partition A in non-empty subsets of cardinality 3, 1 and 1 or 2, 2 and 1. So, we have $\binom{5}{3} = 10$ possibilities in the first case and $\binom{5}{2,2,1}/2 = 15$ in the second case.

Theorem 10: `The analog of Pascal's Identity for Stirling numbers is`

$$\boxed{\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}}. \tag{19}$$

Proof: We show that the sides count the cardinality of the same set: the set of k-partitions of a set $A = \{1,2,3,\ldots,n\}$ with n distingushable elements, in which the partition sets are non-empty and the order is irrelevant (exactly as in the definition of $\left\{ {n \atop k} \right\}$). Hence, the left hand side is basically from the definition of $\left\{ {n \atop k} \right\}$.

Let's divide these partitions into two classes. The first class is the one of those partitions containing n as a single element set: $\{n\}$. In this case, the rest of the partition is a partition of $A \setminus \{n\}$ with only $k-1$ sets. Hence, the number of these partitions is equal to $\left\{ {n-1 \atop k-1} \right\}$ which is the first term in (19) . The second class is the one of those partitions where n appears together with other elements of A. If we remove it, we obtain a partition of $A \setminus \{n\}$ with still k non-empthy subsets. But each one of these, the number of which is $\left\{ {n-1 \atop k} \right\}$, gives k more partitions of the second type because we can add n in k different places. So, we get a total of $k\left\{ {n-1 \atop k} \right\}$ for this type. Therefore the formula (19) must be valid. ■

Some particular cases for computing the Stirling number are easy to establish. It is clear that $\left\{ {n \atop 1} \right\} = 1$. We have $\left\{ {n \atop n} \right\} = 1$ since every subset has to have only one element in it and the order is irrelevant. Also, we have

$$\boxed{\left\{ {n \atop n-1} \right\} = \binom{n}{2}}$$

since the cardinality of the sets can only be 2, 1, 1, ..., 1 wih $n-2$ ones. So, all it counts is what we choose for the 2-element subeset, and that is $\binom{n}{2}$ in possibilitites.

Another simple formula is

$$\boxed{\left\{ {n \atop 2} \right\} = 2^{n-1} - 1}$$

since we can think of the partition as S and $A \setminus S$ where S is every non-empty subset of A. The total number of subsets is 2^n. We have to exclude $S = A$ or $S = \emptyset$, that is $2^n - 2$ and we have to divide by 2 because

every partition is counted twice, once for $S = B$ and once for $S = A \setminus B$, where B an arbitrary non-empty subset of A.

As in the example above we can do the argument in general and obtain

$$\boxed{\left\{ {n \atop n-2} \right\} = \frac{n(n-1)(n-2)(3n-5)}{4!}},$$

which is the result of simplifying $\binom{n}{3} + \binom{n}{2}\binom{n-2}{2}/2$.

A good excercise is to show that

$$\boxed{\left\{ {n \atop 3} \right\} = \frac{3^{n-1} - 2^n + 1}{2}}, \ n \geq 3 \ \text{ or } \ \boxed{\left\{ {n \atop 4} \right\} = \frac{4^{n-1} - 3^n + 3 \cdot 2^{n-1} - 1}{3!}},$$

by induction on n using the recursion formula (19).

We can generate a similar triangle, as the Pascal's triangle using the recurrence rule in (19):

$$\begin{gathered}
\{1\} \\
\{1,1\} \\
\{1,3,1\} \\
\{1,7,6,1\} \\
\{1,15,25,10,1\} \\
\{1,31,90,65,15,1\} \\
\{1,63,301,350,140,21,1\} \\
\{1,127,966,1701,1050,266,28,1\} \\
\{1,255,3025,7770,6951,2646,462,36,1\} \\
\{1,511,9330,34105,42525,22827,5880,750,45,1\}
\end{gathered}$$

3 Inclusion Exclusion Principle

- Combinatorics - the mathematical field of counting
- Partition - a set of one or more nonempty subets of $A : A_1, A_2, \ldots$ such that every element of A is in exactly one set.
- $N(\overline{c_1}\overline{c_2}) = \#$ of elements that do not satisfy (condition) c_1 and do not satisfy c_2.
- two events A and B are independent if $P(A \cap B) = P(A)P(B)$.
- Recurence Relation: A relation that allows the calculation or construction of a term in a sequence, using some (or all) of the previous terms.

- Bayes' theorem for conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad \text{if} \quad P(B) \neq 0$$

- The number of onto functions defined on a set with m elements into a set with n elements is equal to

$$\sum_{i=0}^{n} (-1)^n \binom{n}{i} (n-i)^m.$$

- PIE for an arbitrary number of sets $\{A_i\}_{1\le i\le n}$: Let $S_1 = \sum_{i=1}^{n} |A_i|$, $S_2 = \sum_{i<j}^{n} |A_i \cap A_j|$, $\cdots$, $S_n = |A_1 \cap A_2 \cdots A_n|$. Then,

$$|A_1 \cup A_2 \cup \cdots A_n| = S_1 - S_2 + S_3 + \cdots + (-1)^{n-1} A_n.$$

- (Linear Recurrent Sequence of order I) If $x_{n+1} = ax_n + b$ $(a \neq 1)$ then the gerneral formula of $\{x_n\}$ is given by $x_n = Aa^n - \frac{b}{a-1}$ where A is determined from the initial contitions. If $a = 1$, i.e., $x_{n+1} = x_n + b$, then $x_n = nb + A$.

The PIE for two, three and four sets written explicitly:

- 2 sets:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

- 3 sets:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

- 4 sets:

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3|$$
$$-|A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| +$$
$$|A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

The PIE for two, three and four sets explicitly in its negation form, i.e, the cardinality of the set

$$\overline{A_1} \cap \overline{A_2} \cap \cdots \overline{A_n},$$

where $\overline{A_i}$ measn the complemetary set (in some universe U) of the set A_i.

We let N be is the number of all elements in the universe, and c_i means the condition that an element belongs to the set A_i.

- 2 sets:

$$N(\overline{c_1}\overline{c_2}) = N - N(c_1) - N(c_2) + N(c_1c_2)$$

where $N(c_1c_2)$ is the number of elements in U which satisfy c_1 and c_2.

- 3 sets:

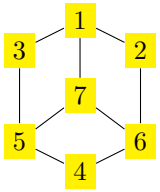
$$N(\overline{c_1}\overline{c_2}\overline{c_3}) = N - N(c_1) - N(c_2) - N(c_3) + N(c_1c_2) + N(c_1c_3) + N(c_2c_3) - N(c_1c_2c_3)$$

- 4 sets:

$$N(\overline{c_1}\overline{c_2}\overline{c_3}\overline{c_4}) = N - N(c_1) - N(c_2) - N(c_3) - N(c_4) + N(c_1c_2) + N(c_1c_3) +$$
$$N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4) - N(c_1c_2c_3) - N(c_1c_2c_4) - N(c_1c_3c_4) - N(c_2c_3c_4) + N(c_1c_2c_3c_4)$$

4 Recurrent Sequences

Some Problems on recursion and PIE



I hear babies crying, I watch them grow
They'll learn much more than I'll ever know
And I think to myself,
'What a wonderful world'
Yes I think to myself,
'What a wonderful world'

Problem 1: *The sequence x_n is given by the recurrence*

$$x_{n+1} = 4x_n - 5x_{n-1} + 2x_{n-2}, \quad n \geq 2,$$

knowing that $x_0 = 1$, $x_1 = -1$ and $x_2 = 2$. Find an exact formula for this sequence .

Solution: (Method I:) We solve the characteristic equation associated to the recurrence: $t^3 - 4t^2 + 5t - 2 = 0$ which can be factored as $(t-1)^2(t-2) = 0$. Hence the sequence is given by $x_n = A + Bn + C2^n$. Because, $x_0 = A + C = 1$, $x_1 = A + B + 2C = -1$ and $x_2 = A + 2B + 4C = 2$, we have to solve this system of three equations with three unknowns. From the first equation $C = 1 - A$ and substituting into the other two gives $B - A = -3$ and $2B - 3A = -2$. From these equations $B = A - 3$ and so $2B = 3A - 2 = 2A - 6$ which implies $A = -4$. Then $B = -7$ and $C = 5$. We then get the closed formula for the general term

$$\boxed{x_n = 5 \cdot 2^n - 7n - 4, \ n \geq 2.} \tag{20}$$

(Method II) If we do the substitution $y_n = x_n - x_{n-1}$ the recurrence relation becomes

$$x_{n+1} - x_n = 3x_n - 3x_{n-1} - 2x_{n-1} + 2x_{n-2} \ \text{ or } \ y_{n+1} = 3y_n - 2y_{n-1}.$$

Now we make another substitution $z_n = y_n - y_{n-1}$:

$$y_{n+1} - y_n = 2y_n - 2y_{n-1} \ \text{ or } \ z_{n+1} = 2z_n.$$

The last recurrence implies $z_n = z_2 2^{n-2}$. We can determine $z_2 = y_2 - y_1 = (x_2 - x_1) - (x_1 - x_0) = (2+1) - (-1-1) = 5$. So, $z_n = 5 \cdot 2^{n-2}$. Then

$$z_2 + z_3 + \cdots + z_n = (y_2 - y_1) + (y_3 - y_2) + \cdots + (y_n - y_{n-1}) = y_n - y_1 \implies$$
$$y_n = y_1 + 5(1 + 2 + \cdots + 2^{n-2}) = -2 + 5(2^{n-1} - 1) = 5 \cdot 2^{n-1} - 7.$$

One more similar step gives x_n

$$y_1 + y_2 + \cdots + y_n = (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) = x_n - x_0 \implies$$
$$x_n = x_0 + 5(1 + 2 + 2^+ \cdots 2^{n-1}) - 7n = 1 + 5(2^n - 1) - 7n = \boxed{5 \cdot 2^n - 7n - 4}.$$

which the same as in (21). □

Problem 2: *(a) Apply inclusion-exclusion principle to find the cardinality of the set*

$$Sur_{6,5} := \{f | f : \{1,2,3,4,5,6\} \to \{a,b,c,d,e\}, \ f \text{ is a surjection}\}.$$

Solutions: (a) Let $A_a = \{f|f : [6] \to \{b,c,d,e\}\}$, $A_b = \{f|f : [6] \to \{a,c,d,e\}\}$, $A_c = \{f|f : [6] \to \{a,b,d,e\}\}$, $A_d = \{f|f : [6] \to \{a,b,c,e\}\}$, and $A_e = \{f|f : [7] \to \{a,b,c,d\}\}$. Each one of these has cardinality 4^6. The intersection of every two has cardinality 3^6, the intersection of every three has cardinality 2^6 and the intersection of every four has cardinality 1^6. Hence, we have (using the complementary version of IEP)

$$|Sur_{6,5}| = 5^6 - 5(4^6) + \binom{5}{2}(3^6) - \binom{5}{3}(2^6) + 5(1) = 1800$$

Remark: If we divide the number obtained by 5! we get the so called Striling number of second kind $S_{6,5} = Sur_{6,5} = 15$ which is the numebr of ways to distribute 6 people into 5 non-empty indistinguishabe offices (One has to be ocupied by two of them and the rest have to be a single ocupancy and this can be done in $\binom{6}{2} = \frac{6\cdot 5}{2} = 15$ indeed.)□

5 Graphs

1. *What is a graph ?*

A graph $G = (V, E)$ is a pair of two sets: V called the set of vertices (or nodes-assumed non-empthy) and E the set of edges (which can be empty), such that E is a multiset with elements of $V \times V$ (for oriented graphs) or $A \cup \{\{v_1, v_2\}|v_1, v_2 \in A\}$ (for undirected graphs). Any $(a, a) \in E$ is called a loop. G is called a simple graph if E is a set and there are no loops (no multiple edges). It is possible to have a mixture of the two situations in which case G is called a mixed graph like the one in Figure 1.

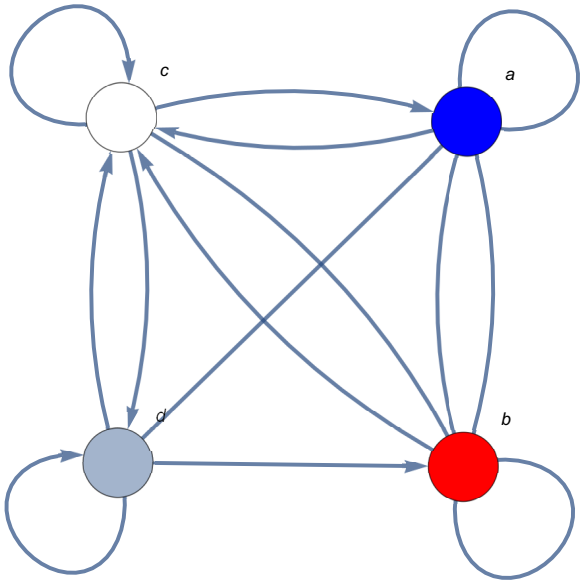


Figure 1, V={a,b,c,d}, E={$a\leftrightarrow b$,$a\leftrightarrow b$,$a\leftrightarrow a$,$c\leftrightarrow b$,$a\to c$,$b\leftrightarrow b$,$c\to a$,$b\to c$,$c\to c$}

The figure above was obtained with Mathematica using the command

Graph[$\{a \leftrightarrow b, a \leftrightarrow b, a \leftrightarrow a, c \leftrightarrow b, a \to c, b \leftrightarrow b, c \to a, b \to c, c \to c\}$, $VertexLabels \to Automatic$,VertexStyle $\to \{a \to Blue, b \to Red, c \to White\}$, EdgeStyle $\to$ Thick, VertexSize $\to$ Medium]

We are mostly interested in simple, undirected graphs (no multiple edges, no loops and there is no order of the vertices of an edge) and finite (V is a finite set). The following table is from our textbook and shows the various terminology when it comes to graphs.

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

2. *What is a sub-graph of a graph ?*

A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ so that $V' \subset V$ and $E' \subset E$.

Notice that asking G' to be a graph in itself, it is an important requirement here. For example, $G' = (\{a, b\}, \{b \to c\})$ is not a subgraph of the graph in Figure 1 (although $V' \subset V$ and $E' \subset E$) because it is not a graph in the first place.

3. *What is an induced graph of a graph ?*

A graph $G' = (V', E')$ is an induced graph of $G = (V, E)$ if (remember that every definition is an if and only if statement, but we are not usually always emphesizing that), $V' \subset V$ and E' is defined by every edge in E that is incident to the points it connects that are in E'.

For example, $G' = (\{a, b\}, \{a \leftrightarrow b, a \leftrightarrow b, a \leftrightarrow a, b \leftrightarrow b\})$ is an induced graph of G in Figure 1 (see Figure 2).

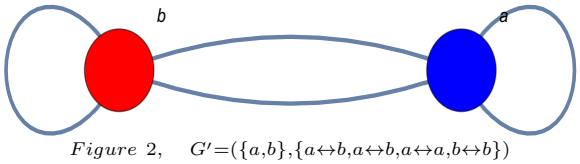


Figure 2, $G'=(\{a,b\},\{a\leftrightarrow b,a\leftrightarrow b,a\leftrightarrow a,b\leftrightarrow b\})$

4. *What is a complete graph ?*

A graph $G = (V, E)$ is called a complete graph on n vertices ($n = |V|$) if G is a simple and finite graph, and E is defined by the property that for every two vertices in V there exists an edge in E which is incident to those two points. As a notation, since this graph depends essentially only on $|V| = n$, it is denoted by K_n.

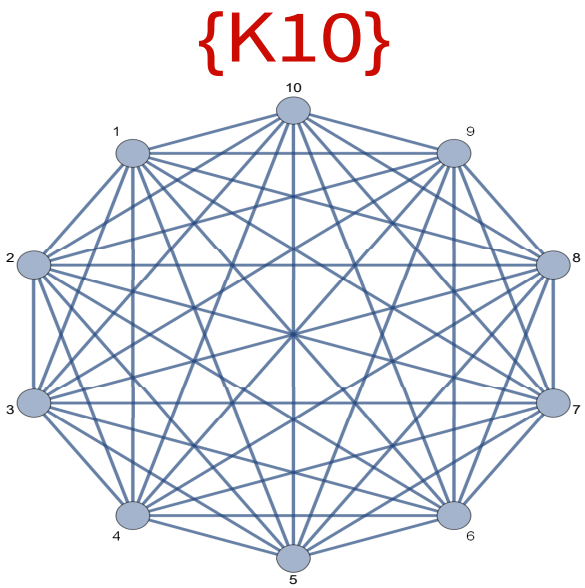


Figure 3, K_{10}

The graph in Figure 3 is K_{10}.

5. *What is a connected graph ?*

A graph $G = (V, E)$ is connected if for every two vertices a and b in V, there exists a sequence of vertices $v_1 = a$, v_2, ..., $v_n = b$ such that there is an edge incident to v_i and v_{i+1} for every $i \in \{1, 2, \cdots, n-1\}$.

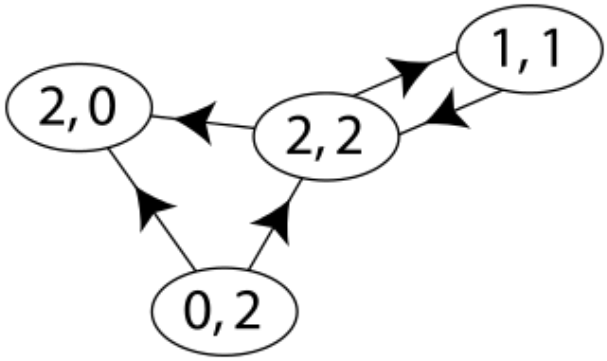
In the case of a directed graph, (v_i, v_{i+1}) must be in E for every $i \in \{1, 2, \cdots, n-1\}$.

6. *When are two graphs are isomorphic?*

Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there is bijection f from V to V' such that if ab is an edge in E, then $f(a)f(b)$ is an edge in E', and vice-versa if $f(a)f(b)$ is an edge in E' then ab is an edge in E.

7. *What is the degree of a vertex ?*

The degree of a vertex v in a finite simple undirected graph $G = (V, E)$ is equal to the number of edges incident to v (loops count twice). We will use the notation $deg_G(v)$ for the degree of v in the graph G. If the graph is directed we have an out-degree and and in-degree, odenoted $deg^+(v)$ and $deg^-(v)$ respectively. The out-degree of a vertex v refers to the number of arcs incident from v. That is, the number of arcs directed away from the vertex v. That is, the number of arcs directed away from the vertex v. The in-degree of a vertex v refers to the number of arcs incident to v.



For example, the degree of a in the graph in Figure 1 (when the direction of arcs is ingnored) is $deg_G(a) = 6$. In the case of a directed graph, as in the one above, the vertices are labeled by the in-degree and out-degree.

8. *Is it possible to have a (simple) graph with* 6 *vertices and degrees 4,3,2,1,1,1 ?(explain your answer)*

In Figure 4, we have a graph G with 6 vertices ($V = \{1, 2, 3, 4, 5, 6\}$) and the degrees of the vertices are: $d_G(1) = 4$, $d_G(2) = 3$, $d_G(3) = 2$, $d_G(4) = 1$, $d_G(5) = 1$, and $d_G(6) = 1$.

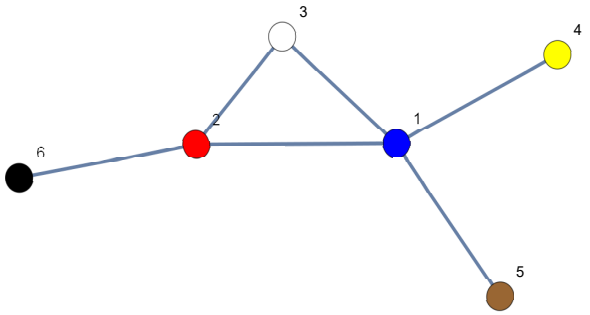


Figure 4, V={1,2,3,4,5,6}, E={1↔2,1↔3,1↔4,1↔5,2↔3,2↔6}

This shows that such a graph, with the given degree restrictions, is possible.

6 Bipartite Graphs and other examples

1. What is an empty graph ?

An empty graph is a graph with no edges.

2. What is a bipartite graph ?

A bipartite graph is a graph whose vertex set can be partitioned into two sets V_1 and V_2 ($V_1 \cap V_2 = \emptyset$, one can be the empty set) such that for every edge $e = uv \in E$ we have $u \in V_1$ and $v \in V_2$.

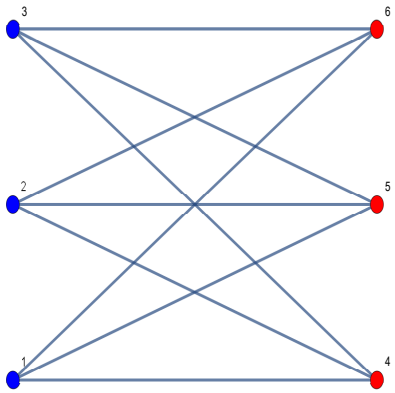


Figure 1, $G=K_{3,3}$

In Figure 1, we have such a bipartite graph in which V_1 are the vertices in blue and V_2 are the red colored ones. We will see that this is actually the so called $K_{3,3}$ because it has all possible edges between vertices in V_1 and vertices in V_2. So, a bipartite graph with this partition of vertices is a subgraph of $K_{3,3}$.

3. What is a complete bipartite graph ?

A complete bipartite graph $G = (V, E)$ is a bipartite graph having the partion of vertices V_1 and V_2 ($V = V_1 \cap V_2$) and it contains all possible edges determined by this partion, i.e., for every vertex u in the set V_1 and every vertex v in V_2 the edge uv is in E.

It is not difficult to show that the complete bipartite graph is determined by the number of vertices in V_1 and V_2. So, if $|V_1| = m$ and $|V_2| = n$, we denote this graph by $K_{m,n}$. In Figure 2 we have a representation of $K_{3,4}$.

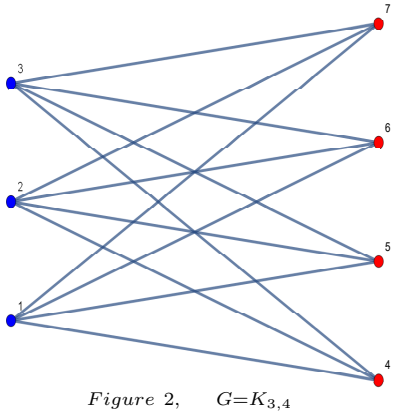


Figure 2, $G=K_{3,4}$

4. What is a path graph and what is a path in a graph ?

A path graph is a (simple, undirected) graph $G = (V, E)$ whose vertices can be arranged in a sequence, $V = \{v_1, v_2, v_3, \cdots, v_n\}$, such that the edge set is $E = \{v_i v_{i+1} | i = 1, \cdots, n-1\}$.

It turns out that the only important information here is the number of vertices in such a graph, and so we have the standard notation for a path graph with n vertices: P_n.

A path in a graph is a sub-graph isomorphic to a path graph.

In the Figure 3, we have a path in the grid graph $Grid[3, 4]$ connecting the vertices 1 and 12. This is actually one of the shortest possible paths connecting the two vertices.

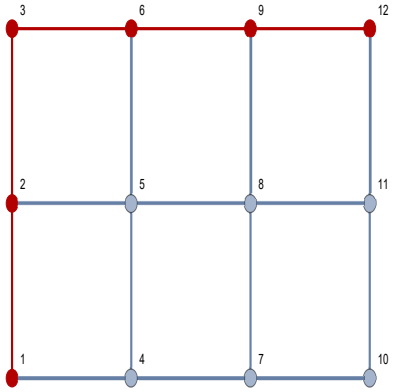


Figure 3, G=Grid[3,4]

5. What is a regular graph ?

Let k be a non-negative integer. A k-regular graph is is a (simple undirected) graph in which every vertex has a vertex degree equal to k.

If $k = 0$ we obtain the empty graph. If $k = 1$, we have a union of P_2's. For $k = 2$, we have a union of cycles. The graph K_n (complete graph on n vertices) is clearly $n - 1$-regular.

The graph in Figure 4 is the only other 3-regular graph (besides a triangular prism shape graph) on six vertices, called the Yutsis graph of the 9-j symbol (wikipedia).

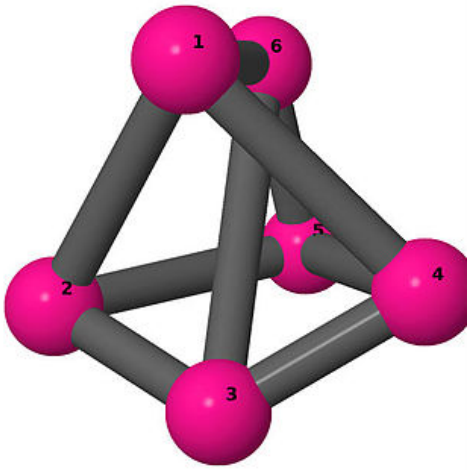


Figure 4, $K_{3,3}$ utility graph

6. What is a cycle graph?

A cycle graph is a (simple, undirected) graph $G = (V, E)$ whose vertices can be arranged in a sequence, $V = \{v_1, v_2, v_3 \cdots v_n\}$ $(n \geq 3)$, such that the edge set is $E = \{v_i v_{i+1} | i = 1, \cdots, n-1\} \cup \{v_n v_1\}$.

The essential information in such a graph is the number of vertices and so there is a standard notation

for the cyclic graph on n vertices: C_n.

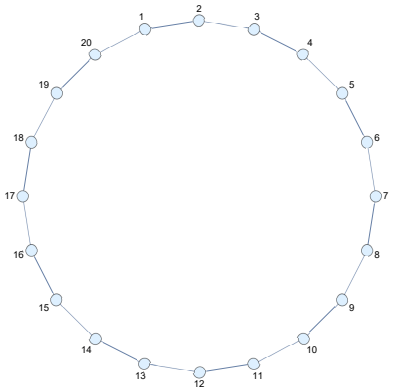


Figure 5, G=C_{20}

In Figure 5 we see a representation of C_{20}.

7. State the Handshaking Theorem (p. 653 in our textbook) and include a proof by induction on the number of edges.

The Handshaking Theorem Let $G = (V, E)$ be a simple undirected graph with m edges and the degree for every vertex is denoted by $deg_G(v)$. Then we have the identity

$$\sum_{v \in V} deg_G(v) = 2m. \tag{21}$$

Proof: We prove this by induction on m.

Basis Step: Let us start with $m = 0$. In this situation there are no edges and so $deg(v) = 0$ for all $v \in V$. Hence, the left hand side of (21) is zero. The same is true for the right hand side of (21).

Inductive Step: Let us assume that (21) is true for some $m \geq 0$. Then let us pick a graph G with $m + 1$ edges. If we delete one edge of G we end up with a graph the has m edges and so we can use the induction hypothesis on that. Let us call the new graph G'. By induction hypothesis,

$$\sum_{v \in V} deg_{G'}(v) = 2m. \tag{22}$$

Now, the degree of every vertex in G' is the same as the degree in G, except for the two vertices which are the endpoints of the edge we deleted. For those we have $deg_G(v) = deg_{G'}(v) + 1$ and so the contribution of those vertices in (22) is a 2. We conclude that

$$\sum_{v \in V} deg_G(v) = 2 + \sum_{v \in V} deg_{G'}(v) = 2 + 2m = 2(m + 1),$$

which is exactly (21) for a graph with $m + 1$ edges.

By the PMI, we conclude that (21) is true for any graph. □

8. What is the characterization of bipartite graphs that is suggested in the videos for bipartite graphs in terms of coloring?

A graph is bipartite if and only if it can be (vertex) properly colored with only two colors. (properly colored means no edge is incident to vertices of the same color).

9. In the figure below you have two cubic graphs on 8 vertices which are not isomorphic. Give another example of two cubic graphs with 10 vertices which are not isomorphic.

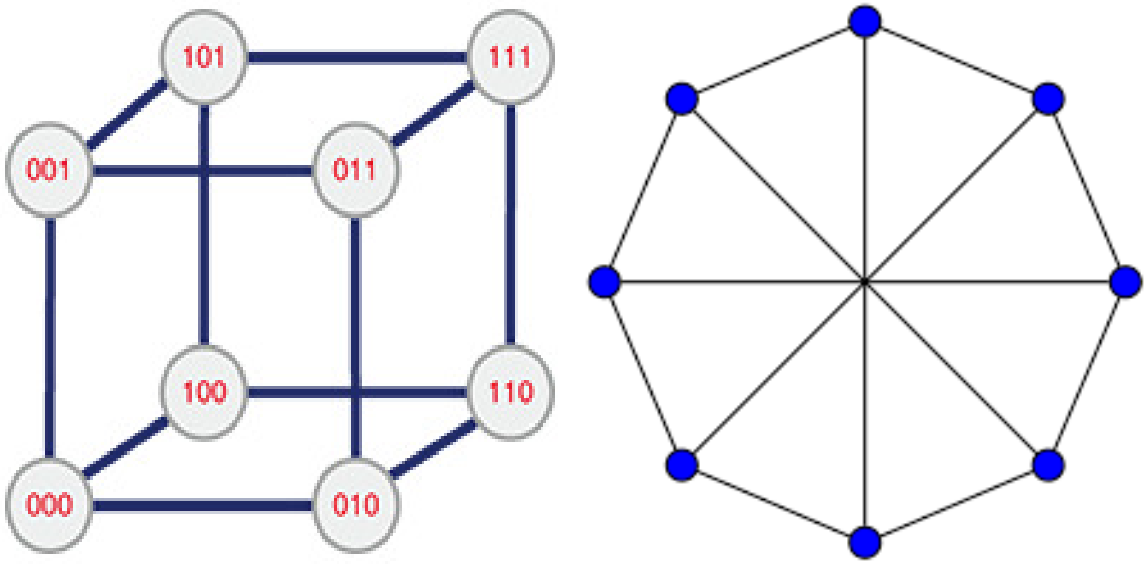


Figure 6, Q_3 and Wagner Graph

The graphs are not isomorphic because one is bipartite and the other on is not (one has odd cycles and the other is not).

Another example is included in Figure 7: Pentagonal prism graph and the famous Petersen Graph.

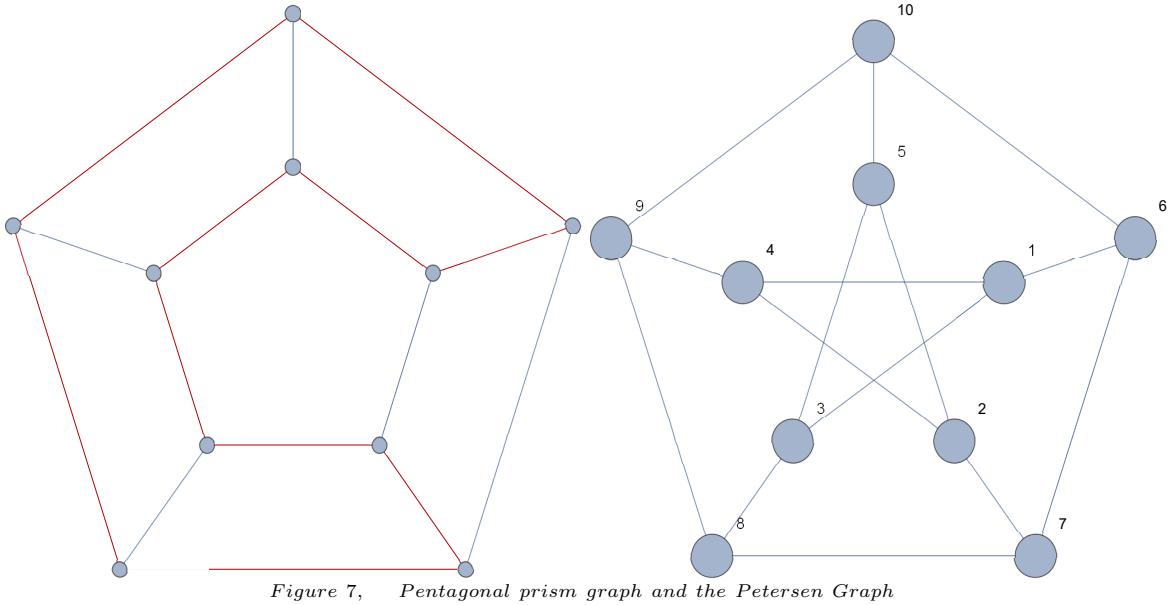


Figure 7, Pentagonal prism graph and the Petersen Graph

They are not isomorphic because one has four cycles and the other one it does not.

7 Eulerian and Hamiltonian Graphs

1. What is a walk in a graph ?

A walk in a graph $G = (V, E)$ is a sequence of vertices v_1, v_2, v_3, ..., v_n (from V), such that $v_i v_{i+1} \in E$ for all $i = 1, 2, ..., n-1$.

The vertices v_i may repeat in this definition. So a walk in the cycle C_3 may be to go around the cycle twice. We say that the walk connects $u = v_1$ with $v = v_n$.

2. What is the definition of a trail ?

A trail is a walk with no repeated edges.

In a trail one can repeat vertices. In particular, one can have $v_1 = v_n$. In this case the trail is called a circuit.

3. What is a Euler trail ?

A Euler trail is a trail that contains all edges of the graph.
In Figure 1, we can find such a trail :

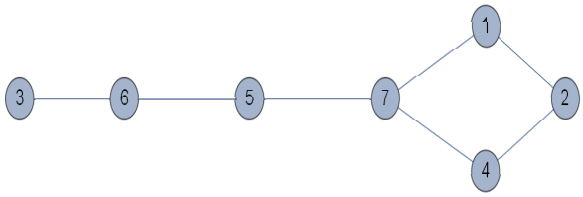


Figure 1,

$$3 \to 6 \to 5 \to 7 \to 1 \to 2 \to 4 \to 7.$$

4. What is Euler tour (our textbook talks about circuits)?

An Euler tour is an Euler trail that starts and ends at the same vertex. An Eulerian graph is a graph that has an Euler tour.

In Figure 2, we can find such a Euler tour :

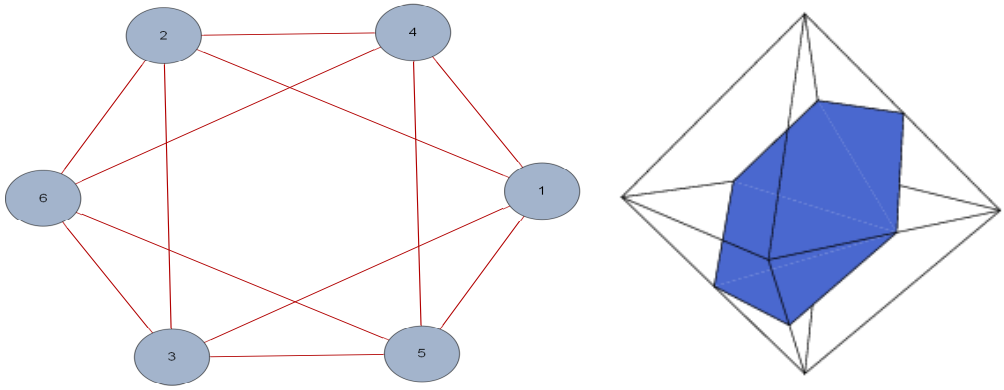


Figure 2, Octahedral Graph

$$1 \to 5 \to 6 \to 4 \to 5 \to 3 \to 6 \to 2 \to 4 \to 1 \to 3 \to 2 \to 1$$

A graph may have a Euler trail and have no Euler tour (the graph in Figure 1).

5. What is a Hamiltonian path/cycle ?

A Hamiltonian path is a path in the specific graph that covers every vertex once. An Hamiltonian cycle is a cycle that covers every vertex once. A graph which has a Hamiltonian cycle is simply called Hamiltonian.

Remarks: Every cycle C_n $(n \geq 3)$ is clearly Hamiltonian. It is known that Petersen graph is not Hamiltonian but it contains a Hamiltonian path (see Figure 3).

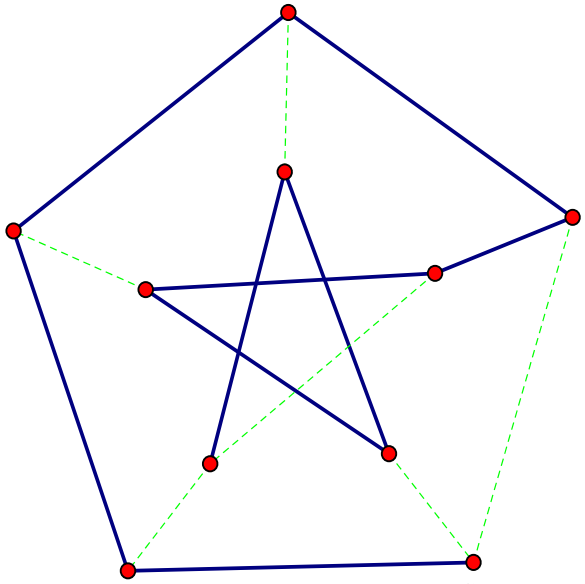


Figure 3, Hamiltonian Path in Petersen's Graph

6. What is the characterization of Eulerian graph?

A graph is Eulerian iff it is connected and every vertex has an even degree.

Remarks: So, as a result, every $2k$-regular connected graph is Eulerian. Such a graph is the one in Figure 2 (Octahedral Graph). If we only want an Eulerian path then the characterization is : "A graph has an Euler trail iff it has at most two vertices of odd degree. "

7. List the problems solved in the videos

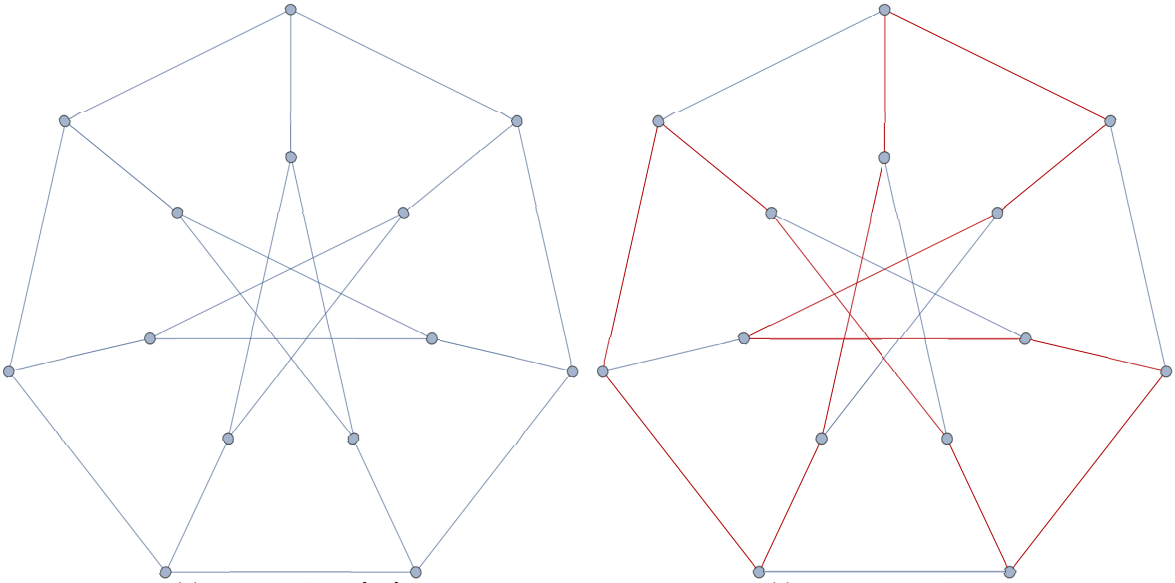
- Show that $K_{m,n}$ is Hamiltonian iff $n = m \geq 2$.
- Give an example of a connected graph that has no Hamiltonian path.
- Can a Hamiltonian path be used to form a Hamiltonian cycle?

8. Our textbook gives some sufficient conditions for a graph to be Hamiltonian. List them here.

Dirac's Theorem If $G = (V, E)$ is a simple graph with n vertices $(n \geq 3)$ such that the degree of every vertex in V is at least $\frac{n}{2}$, then G is Hamiltonian.

Ore's Theorem If $G = (V, E)$ is a simple graph with n vertices ($n \geq 3$) such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u and v in V, then G is Hamiltonian.

9. In the figure below you have the Petersen graph $P[7, 3]$. This graph is Hamiltonian. Find a Hamiltonian cycle.



(a) *PetersenGraph*[7,3] (b) *Hamiltonian Cycle*

Figure 4

A Hamiltonian Cycle is shown in Figure 4(b).

10. In the video no 19 of Dr. Saranda Herke there is another characterization of bipirtite graphs. What is it ?

A graph is bipartite iff it contains no odd cycle.

8 Trees

Definition: A tree is graph which is connected and acyclic (no cycles).

Theorem: A graph is a tree if and only if there is a unique simple path between any two of its vertices.

1. What is a bridge in a graph?

An edge is called a bridge in a graph, if the graph obtained by removing that edge, it has more connected components than the original graph.

In the Figure 1, the edge 57 is such a bridge. None of the other edges is a bridge.

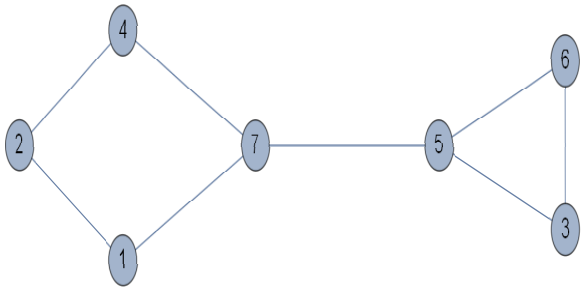


Figure 1, **Edge 57 is a bridge**

2. What is the characterization of a brige in terms of cycles ?

An edge is a bridge if and only if the edge is not part of any cycle of the graph.

The graph in Figure 1 every edge other than 57 is part of a cycle so it cannot be a bridge.

3. What is Cayley's formula about the number of labeled trees on n vertices ?

Cayley's Theorem: *There are n^{n-2} labeled trees on n nodes.*

4. What is the characterization of a tree in terms of the number of edges and the number of vertices?

A graph $G = (V, E)$ is a tree iff G is acyclic and $|E| = |V| - 1$.
In Figure 2 we have an example of a tree on 13 vertices. So, it must have 12 edges.

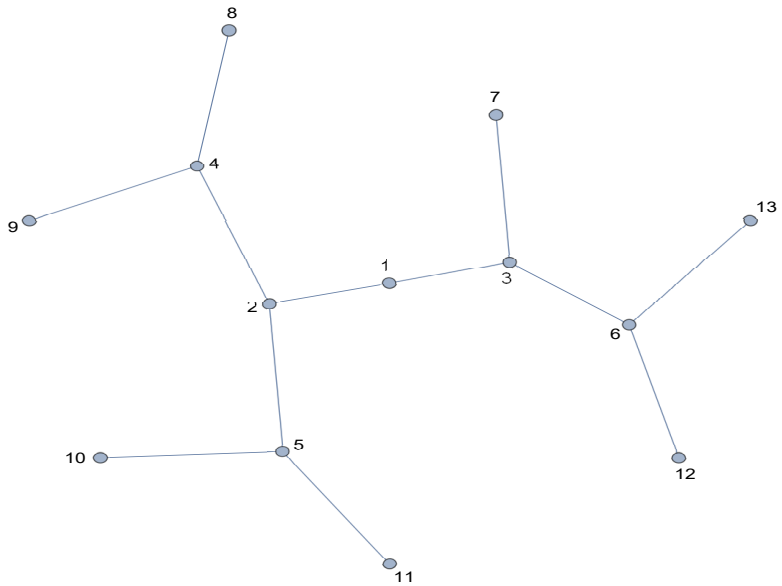


Figure 2, **Example of a tree, call it H**

5. Give the defintion of a spanning tree of a graph?

A spanning tree of a graph G is a subgraph of G that is a tree and it is spanning, i.e., it contains all of the vertices of G.

If we add any number of edges to the graph in Figure 2, we get a graph G so that H is a spanning tree

of G (see Figure 3)

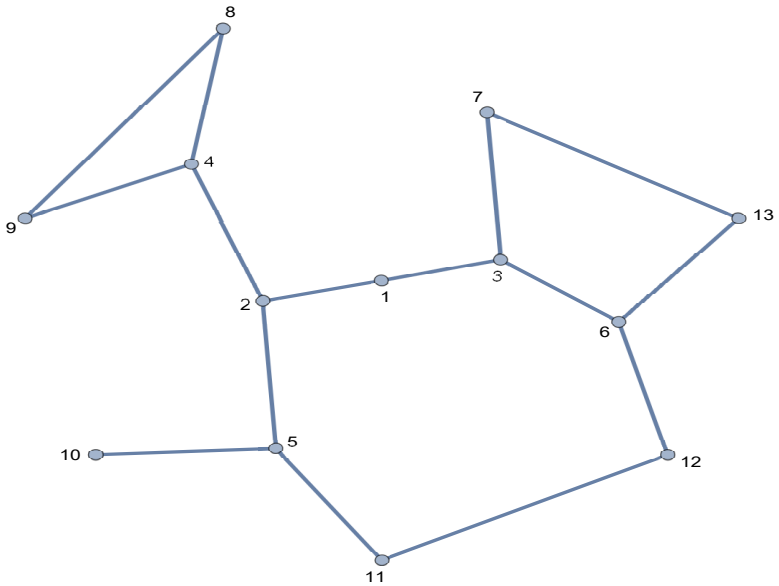


Figure 3, **Graph G**

6. What are the three ways to characterize a tree ?

A graph $G = (V, E)$ is a tree iff it has any two of the following properties:

- *The graph is connected.*
- *The graph is acyclic.*
- $|E| = |V| - 1$

7. List the types of trees Dr. Saranda Herke is talking about in the last video.

She is talkintg about three types of trees:

1. *A path (P_n)*
2. *A star, the complete bipartite graph of $K_{1,n}$ for some $n \geq 1$.*
3. *A double star, two stars connected by an edge that is incident to the centers of each one of the stars.*
4. *A catepillar, which is a tree in which all the vertices are within distance one of a central path.*

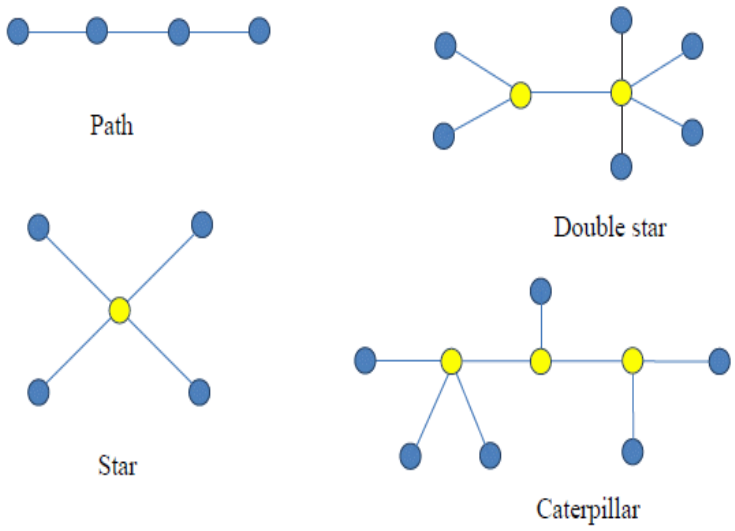


Figure 4, **Graph G**

8. Our textbook gives the definitions of internal vertices of a tree, m-ary tree and a full m-ary tree. What are those?

Definition: A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

"The terminology for trees has botanical and genealogical origins. Suppose that T is a rooted tree. If v is a vertex in T other than the root, the parent of v is the unique vertex u such that there is a directed edge from u to v. When u is the **parent** of v, v is called a child of u."

Internal vertices (in a tree with at least three vertices) are vertices that have children, i.e., a vertex of a degree at least two. An internal vertex is also called a branch vertex.

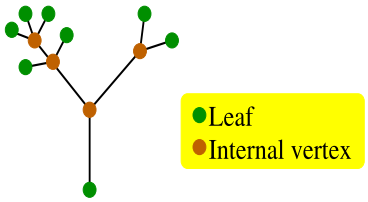
An m-ary tree is a tree in which every internal vertex has no more than m children, i.e., the degree is not more than m.

A full m-ary tree is a tree where every internal vertex has exactly m children.

Theorem: A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.

Remark: A rooted tree is a tree in which one particular vertex has been designated a root. All the other vertices are called decendents of the root. It turns out that there is a unique path from any two vertices in a tree, so in particular, there is a unique path from the root to a fixed vertex. The length of that path is usually called the height of the vertex.

Leaves



- A vertex of degree 1 is called a *leaf*.
  This tree has 8 leaves (including the bottom vertex).
- Sometimes, vertices of degree 0 are also counted as leaves.
- A vertex with degree $\geqslant 2$ is an *internal vertex*.
  This tree has 4 internal vertices.

Prof. Tesler Ch. 10.1: Trees Math 184A / Winter 2017 4 / 15

Figure 4, **Some examples**

9. In the figure below you have the complete binary tree on 3 levels (Mathematica command CompleteKaryTree[4, 2]). Include a figure with complete 3-ary tree on 4 levels. What is the number of vertices of a complete 3-ary tree on n levels?

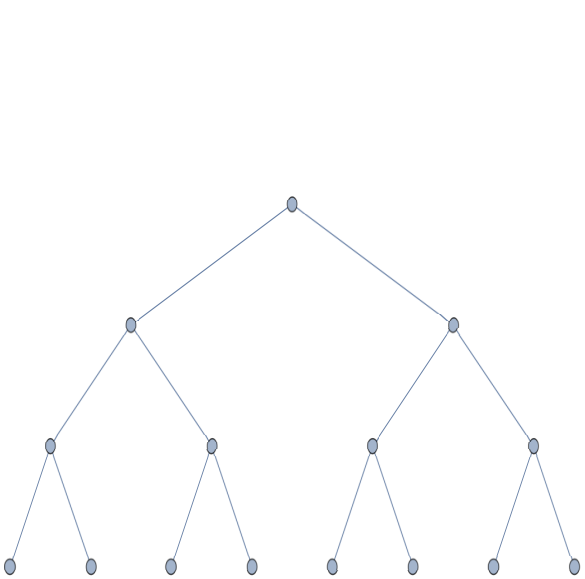


Figure 5(a), *CompleteKaryTree*[4,2]

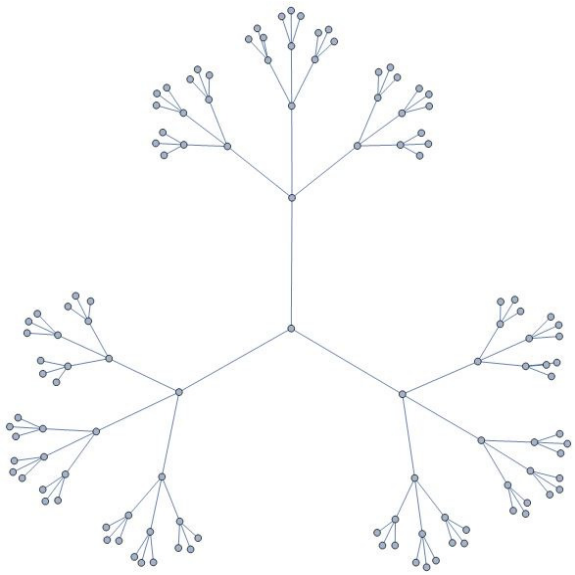


Figure 5(b), *CompleteKaryTree*[5,3]

The formula for the number of vertices of a complete 3-ary tree on n levels is

$$\boxed{|V| = \sum_{k=0}^{n} 3^k = (3^{n+1} - 1)/2.}$$

In the case of Figure 5(b) we get $(3^5 - 1)/2 = 121$ vertices.

10. The textbook has a few more concepts about balanced m-ary tree and the height of a rooted tree. Include these definitions here and the result about the number of leaves and the height of a full and balanced m-ary tree.

The height of a rooted tree is defined as the length of the longest path from the root to any vertex and it is usually denoted by h. The height of a vertex in a rooted tree is defined as the length of the path from the root to that vertex. A balanced m-ary tree is a m-ary tree in which all of its leaves have a height of h or $h - 1$.

Theorem: *There are at most m^h leaves in an m-ary tree of height h.*

Corollary: *If an m-ary tree of height h has l leaves, then $h \geq \lceil \log_m l \rceil$. Furthermore, if the m-ary tree is full and balanced, then $h = \lceil \log_m l \rceil$.*

The ceiling function, $x \to \lceil x \rceil$, is the smallest integer greater than or equal to x.

Problem (Rosen #24, page 756): Either draw a full m-ary tree with 76 leaves and height 3, where m is a positive integer, or show that no such tree exists.

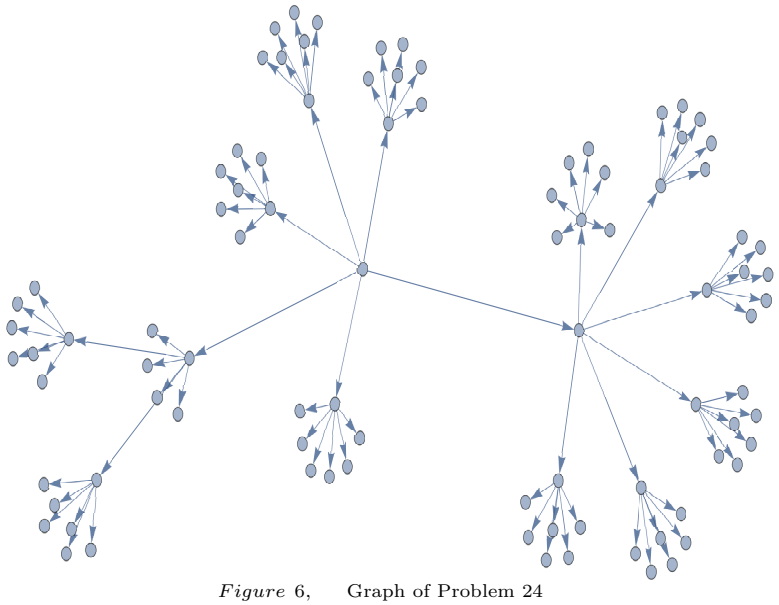


Figure 6, Graph of Problem 24

9 Coloring Graphs, Chromatic Number and Chromatic Polynomial

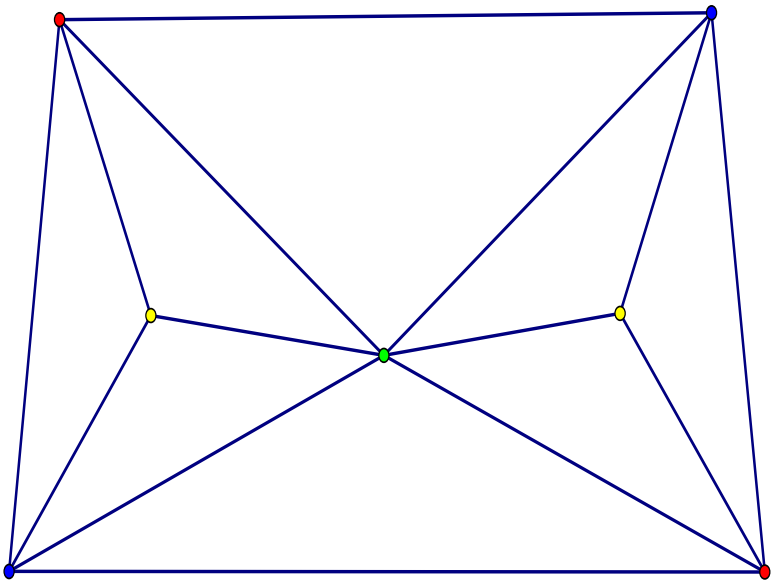


Figure 1, Proper coloring of a graph

Definition: *A proper coloring of a graph $G = (V, E)$ is a map $C : V \to \{1, 2, ...n\}$ such that for every edges $e = ab \in E$, $C(a)$ and $C(b)$ are distinct. THe smallest n for which a proper coloring exists is called the chromatic number of the graph G and it is denoted by $\chi(G)$.*

The graph in Figure 1, has a proper coloring with four colors and one can see that 3 colors are not enough. This means that $\chi(G) = 4$.

In fact, we have the following simple observations:

- $\chi(K_n) = n$ by giving every vertex a different color. No two vertices in K_n can be colored the same way because there is an edge incident to any given two vertices.
- For a subgraph H of a graph G, we have $\chi(H) \leq \chi(G)$.
- $\chi(C_n) = \begin{cases} 2 \text{ if } n \text{ is even,} \\ \\ 3 \text{ if } n \text{ is odd.} \end{cases}$
- $\chi(P_n) = \begin{cases} 2 \text{ if } n \text{ is even,} \\ \\ 3 \text{ if } n \text{ is odd.} \end{cases}$
- $\chi(P_n) = 2$ (n is the number of edges)
- $\chi(G) = 1$ iff G is the null graph

The following is a very famous theorem.

Theorem: Every planar graph can be properly colored with four colors.

A graph is *planar* if it can be represented in the plane with no edge crossings. The graph in Figure 1 is planar and it contains K_4 as subgraphs.

Another important theorem is the next one and we have already talked about it.

Theorem: A graph (which is not the null graph) is bipartite iff $\chi(G) = 2$.

As a practical application of proper coloring of a graph is the problem of scheduling final exams. There are algorithms for determining the chromatic number of a graph. One such method is the use of the so called chromatic polynomial.

9.1 Chromatic polynomial

Definition: *The chromatic polynomial of a graph is a polynoimal $\chi_G(k)$ which counts the number of different proper coloring using k colors.*

Let us look at some examples. Suppose we take $G = K_n$. Having k colors, say we start coloring the vertices of K_n in order. The first vertex can be colored in k ways. The second can be colored in $k-1$ ways since we have used one of the colors. This continues until we finish with all vertices, and we see that $\chi_{K_n} = k(k-1)(k-2)\cdots(k-n+1)$. We observe that $k = n$ is the smallest value that makes this polynomial non-zero. This is actually the idea in general:

$$\chi(G) = min\{k|\chi_G(k) > 0\}.$$

Let's look at another example. Suppose that $G = P_n$. We color the first vertex (of order 1) in k ways. Then the second one adjacent to the first in $k-1$. Then the third is also $k-1$ becuase we only haev to stay away of the previous color used. Then we observe that $\chi_{P_n} = k(k-1)^n$. We see that 2 is the first value of k for which this is not zero. In fact, one can see that any tree on $n+1$ vertices has the same chromatic polynomial as the path graph P_n.

The null graph has thhe cromatic polynomial $\chi_{Null} = k^n$.

One example more trickier is $G = C_n$. Let us say we have to do $n = 5$, and $G = (V, E)$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{1 \leftrightarrow 2, 2 \leftrightarrow 3, 3 \leftrightarrow 4, 4 \leftrightarrow 5, 5 \leftrightarrow 1, \}$. Then we have k possibilities for coloring vertex 1, $k-1$ possibilities for vertex 2, $k-1$ for vertex 3, $k-1$ for vertex 4, and for vertex 5 let's say we pick all possibilites of $k-1$ to stay away from the color of 4. That is $k(k-1)^3$ and weneed to subtract to situations in which 4 and 5 have the same color, $k(k-1)^3$ and take away $k(k-1)(k-2)$. The polynomial is

$$\chi_{C_5}(k) = k(k-1)^4 - [k(k-1)^3 - k(k-1)(k-2)] = k(k-1)(k-2)(k^2-2k+2).$$

The smallest integer k that makes this non-zero is clearly $k = 3$, so $\chi(C_5) = 3$ as we observed before.

This kind of strategy can be generalized. We have the following theorem.

Theorem (Deletion-contraction formula): Suppose that G is a simple graph and $e = ab$ is some edge. Denote by $G - e$ the graph with the edge e deleted and by $G \setminus e$ the graph obtained by identifying the points a and b (called the contraction by edge e). Then, we have

$$\chi_G = \chi_{G-e} - \chi_{G\setminus e}.$$

Example 1: Let take the graph $G = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = E = \{1 \leftrightarrow 2, 2 \leftrightarrow 3, 3 \leftrightarrow 1, 2 \leftrightarrow 4\}$. Using this formula we see that if take the edge $e = 1 \leftrightarrow 2$, we get

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G\setminus e}(k) = k(k-1)^3 - k(k-1)^2 = k(k-1)^2(k-2).$$

This tells us that $\chi(G) = 3$.

Example 2: Let us consider the graph G in Figure 2 below:

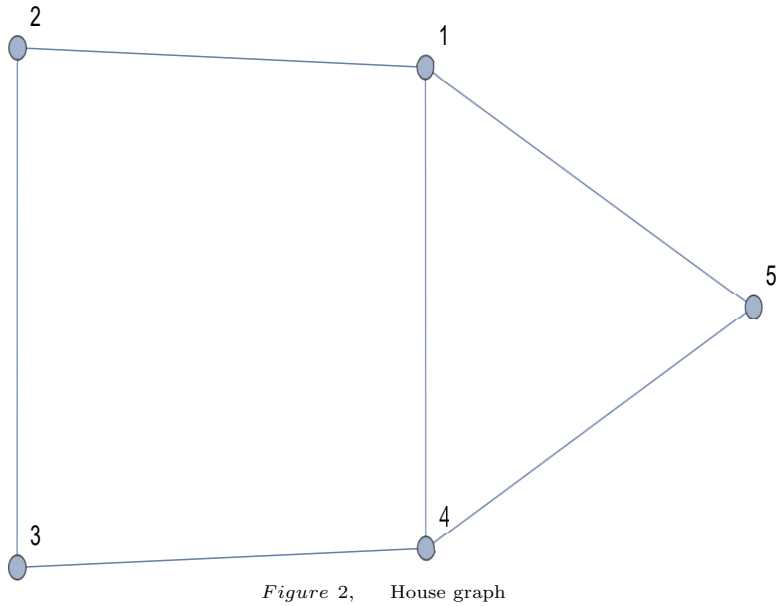


Figure 2, House graph

If we pick the edge $e = 14$, the graph $G - e$ is the C_5 and $C \setminus e$ is the graph in Example 2. So, we have

$$\chi_G(k) = k(k-1)(k-2)(k^2-2k+2) - k(k-1)^2(k-2) = k(k-1)(k-2)(k^2-3k+3).$$

In this case $\chi(G) = 3$.
Homework: Show that $\chi(Petersen\ Graph) = 3$.