Problems solved or proposed from the MAA Journals and other publications

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Problem 1. Let n be a positive integer, and let f be a continuous real-valued function on $[0,1]$ with the property that $\int_0^1 x^k f(x)dx = 1$ for $0 \le k \le n-1$. Prove that $\int_0^1 (f(x))^2 dx \ge n^2$.

Problem 2. Find the following limit

$$L = \lim_{n\to\infty} \frac{1}{n} \int_0^n \left(\frac{x \ln(1+x/n)}{1+x} \right) dx. \tag{1}$$

Problem 3. Show that for $|x| < 1$,

$$\left(\sum_{n=1}^{\infty} \lfloor \frac{n}{\sqrt{2}} \rfloor x^n \right) \left(\sum_{n=1}^{\infty} x^{\lfloor n\sqrt{3} \rfloor} \right) = \left(\sum_{n=1}^{\infty} \lfloor \frac{n}{\sqrt{3}} \rfloor x^n \right) \left(\sum_{n=1}^{\infty} x^{\lfloor n\sqrt{2} \rfloor} \right), \tag{2}$$

where $\lfloor x \rfloor$ denotes the floor function.

Problem 4. For every nonnegative integer n, evaluate

$$I_n := \int_0^{\infty} \frac{x^n dx}{e^x + \sum_{k=0}^n \frac{x^k}{k!}}.$$

Problem 5. Consider the polynomial

$$f(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4,$$

where a, b, c, and d are positive real numbers. Prove that if f has four positive distinct roots, then $a > b > c > d$.

Problem 6. Evaluate

$$(a) \quad \int_0^1 \frac{\ln(1+x)}{1+x^2}dx, \quad (b) \quad \int_0^1 \frac{\arctan x}{1+x}dx.$$

Problem 7. Let the consecutive vertices for a regular n-gon P be denoted A_0, ..., A_{n-1}, in order, and let $A_n = A_0$. Let M be a point such that for $0 \leq k < n$ the perpendicular projections of M onto each line A_kA_{k+1} lie interior to the segment $\overline{A_kA_{k+1}}$. Let B_k be the projection of M onto $\overline{A_kA_{k+1}}$. Show that

$$\sum_{k=0}^{n-1} Area(\triangle MA_kB_k) = \frac{1}{2}Area(P). \tag{3}$$

Problem 8. Consider a triangle ABC. Let $\mathcal{O}$ be the circle circumscribed to ABC, r the radius of the circle inscribed to ABC, and s the semiperimeter. Let arc(BC) be the arc of $\mathcal{O}$ opposite A, and define arc(CA) and arc(AB) similarly. Let $\mathcal{O}_{\mathcal{A}}$ be the circle tangent to AB and AC and internally tangent to $\mathcal{O}$ along arc(BC), and let R_A be its radius. Define $\mathcal{O}_{\mathcal{B}}$, $\mathcal{O}_{\mathcal{C}}$, R_B, and R_C similarly. Show that

$$\frac{1}{aR_A} + \frac{1}{bR_B} + \frac{1}{cR_C} = \frac{s^2}{rabc}.$$

Problem 9. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous differentiable function such that $\phi(0) = 0$ and ϕ' is strictly increasing. For $a > 0$, let C_a denote the space of all continuous functions from $[0, a]$ into $\mathbb{R}$, and for $f \in C_a$, let $I(f) = \int_0^a (\phi(x)f(x) - x\phi(f(x)))dx$. Show that I has a finite supremum on C_a and that there exists an $f \in C_a$ at which supremum is attained.

Problem 10. Let I_a, I_b, I_c and, r_a, r_b, r_c be respectively the excenters and exradii of the triangle ABC. If ρ_a, ρ_b ρ_c are the inradii of triangles I_aBC, I_bCA and I_cAB, show that

$$\frac{\rho_a}{r_a} + \frac{\rho_b}{r_b} + \frac{\rho_c}{r_c} = 1$$

Problem 11. Determine for which integers a the Diophantine equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a}{xyz}$ has infinitely many integer solutions (x, y, z) such that $\gcd(a, xyz)$=1.

Problem 12. Let f and g be continuous real valued functions on $[0, 1]$. Prove that there exists c in $(0, 1)$ such that

$$\int_0^1 f(x)dx \int_0^c xg(x)dx = \int_0^1 g(x)dx \int_0^c xf(x)dx.$$

Problem 13. Let f be a nonconstant entire function with nonnegative Taylor series coefficients. Prove that $\lim_{r\to\infty} \frac{f(r)}{rf'(r)}$ exists and is rational.

Problem 14. In triangle ABC, let M and Q be points on segment AB, and similarly let N and R be points on AC, and P and S be points on BC. Let d_1 be the line through M and N, d_2 the line through P and Q, and d_3 the line through R and S. Let $\rho(X, Y, Z)$ denote the ratio of the length of XZ to that of XY. Let $m = \rho(M, A, B)$, $n = \rho(N, A, C)$, $p = \rho(P, B, C)$, $q = \rho(Q, B, A)$, $r = \rho(R, C, A)$, $s = \rho(S, C, B)$. Prove that the lines d_1, d_2, and d_3 are concurrent if and only if

$$mpr + nqs + mq + nr + ps = 1.$$

Problem 15. Prove that if H is a finite subgroup of the group G of all continuous bijections of $[0, 1]$ to itself, then the order of H is 1 or 2.

Problem 16. Let $f : [0, 1] \to [0, \infty)$ be a continuous function such that $f(0) = f(1) = 0$ and $f(x) > 0$ for $0 < x < 1$. Show that there exists a square with two vertices in the interval $(0, 1)$ on the x-axis and the other two vertices on the graph of f.

Problem 17. Let f be a twice-differentiable real-valued function with continuous second derivative, and suppose that $f(0) = 0$. Show that

$$\int_{-1}^{1} (f''(x))^2 dx \geq 10 \left(\int_{-1}^{1} f(x) dx \right)^2 .$$

Problem 18. Let f be a continuous real-valued function on $[0, 1]$ such that $\int_0^1 f(x)dx = 0$. Prove that there exists c in the interval $(0, 1)$ such that $c^2 f(c) = \int_0^c (x^2 + x) f(x) dx$.

Problem 19. Given four concentric circles, find a necessary and sufficient condition that there be a rectangle with one corner on each circle.

Problem 20. For which pairs (a, b) of positive integers do there exist infinitely many positive integers n such that n^2 divides $a^n + b^n$?

Problem 21. Let

$$G_n = \prod_{k=1}^{n} \left(\prod_{j=1}^{k-1} \frac{j}{k} \right),$$

and let $\overline{G}_n = 1/G_n$.

(a) Show that if n is an integer greater then 1, then $\overline{G}_n$ is an integer.

(b) Show that for each prime p, there are infinitely many n greater then 1 such that p does not divide $\overline{G}_n$.

Problem 22. Let a , b, and c be the side lengths of a triangle, and let r_a, r_b, and r_c be the corresponding exradii. Prove that

$$\frac{a^2}{r_a^2} + \frac{b^2}{r_a^2} + \frac{c^2}{r_a^2} = 8 \left(\frac{r_a + r_b + r_c}{a + b + c} \right)^2 - 2. \tag{4}$$

Problem 23. Let $f(x) = \frac{x}{\ln(1-x)}$ for $x \in (0,1)$. Prove that for $0 < x < 1$,

$$\sum_{n=1}^{\infty} \frac{x^n(1-x)^n}{n!} f^{(n)}(x) = -\frac{1}{2} x f(x). \tag{5}$$

Problem 24. Let S be an additive semigroup of positive integers. Show that there is a finite subset T of S that generates S and that is contained in every generating set of S.

Problem 25. Given $0 \le a \le 2$, let $\{a_n\}$ be the sequence defined by $a_1 = a$ and $a_{n+1} = 2^n - \sqrt{2^n(2^n - a_n)}$ for $n \ge 1$. Find $S = \sum_{n=1}^{\infty} a_n^2$.

Problem 26. Let C_0, C_1, C_2, C_3, with subscripts taken modulo 4, be circles in the Euclidean plane.

(a) Given for $k \in \mathbb{Z}_4$ that C_k and C_{k+1} intersect with orthogonal tangents, and the interiors of C_k and C_{k+2} are disjoint, show that the four circles have a common point.

(b)* Does the same conclusion hold in hyperbolic and spherical geometry?

Problem 27. Let f be a continuous function f from $[0,1]$ into $[0,\infty)$. Find

$$L(f) := \lim_{n\to\infty} n \int_0^1 \left(\sum_{k=n}^{\infty} \frac{x^k}{k} \right)^2 f(x)dx. \tag{6}$$

Problem 28. Let α be a real number with $\alpha > 1$, and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n\to\infty} u_n = 0$ and $\lim_{n\to\infty}(u_n - u_{n+1})/u_n^{\alpha}$ exists and is nonzero. Prove that $\sum_{n=1}^{\infty} u_n$ converges if and only if $\alpha < 2$.

Problem 29. Let x_1, x_2,...,x_n be distinct points in $\mathbb{R}^3$, and k_1, k_2,...,k_n be positive real numbers. A test object at x is attracted to each of x_1, x_2,...,x_n with a force along the line from x to x_j of magnitude $k_j\|x - x_j\|^2$, where $\|u\|$ denotes the usual Euclidean norm of u. Show that when $n \ge 2$ there is a unique point x^* at which the net force on the test object is zero.

Problem 30. Let C be the ring of continuous functions on $\mathbb{R}$, equipped with pointwise addition and pointwise multiplication. Let D be the ring of differentiable function on $\mathbb{R}$, equipped with the same addition and multiplication. The ring identity in both cases is the function f_1 on $\mathbb{R}$, that sends every real number to 1. Is there a subring E of D, containing f_1, that is isomorphic to C? (the ring isomorphism must carry f_1 to f_1.)

Problem 31. Let $ABCD$ be a convex quadrilateral, and suppose there is a point on the diagonal BD with the property that the perimeters ABM and CBM are equal and the perimeters of ADM and CDM are equal. Prove that $|AB| = |CB|$ and $|AD| = |CD|$.

Problem 32. Let f be a convex function from $\mathbb{R}$ into $\mathbb{R}$ and suppose that

$$f(x+y) + f(x-y) - 2f(x) \leq y^2 \tag{7}$$

for all real x and y.
(i) Show that f f is differentiable.
(ii) Show that for all real x and y, $|f'(x) - f'(y)| \leq |x - y|$.

Problem 33. Let p be a prime congruent to 7 modulo 8. Prove that

$$\sum_{k=1}^{p} \lfloor \frac{k^2+k}{p} \rfloor = \frac{2p^2+3p+7}{6}. \tag{8}$$

Problem 34. Let a and b be real, with $1 < a < b$, and let m and n be real with $m \neq 0$. Find all continuous functions f from $[0, \infty)$ to $\mathbb{R}$ such that for $x \geq 0$,

$$f(a^x) + f(b^x) = mx + n. \tag{9}$$

Problem 35. Let f be a continuous map from $[0,1]$ to $\mathbb{R}$ that is differentiable on $(0,1)$, with $f(0) = 0$ and $f(1) = 1$. Show that for each positive integer n there exists distinct number c_1, c_2,..., c_n in $(0,1)$ such that $1 = \prod_{k=1}^{n} f'(c_k)$.

Problem 36. Let f be a bounded continuous function mapping $[0,\infty)$ to itself. Find the following limit

$$L := \lim_{n\to\infty} n\left(\sqrt[n]{\int_0^\infty f(x)^{n+1}e^{-x}dx} - \sqrt[n]{\int_0^\infty f(x)^n e^{-x}dx}\right).$$

Problem 37. Given a positive real number a_0, let $a_{n+1} = \exp(-\sum_{k=0}^n a_k)$ for $n \geq 0$. For which values of b does $\sum_n a_n^b$ converges?

Problem 38. A signed binary representation of an integer m is a finite list a_0, a_1, ... of elements of the set $\{-1,0,1\}$ such that $\sum a_i 2^i = m$. A signed binary representation is *sparse* if no two consecutive entries in the list are nonzero.

(a) Prove that every integer has a unique sparse representation.

(b) Prove that for all $m \in Z$, every non-sparse signed binary representation of m has at least as many nonzero terms as the sparse representation.

Problem 39. Given a tetrahedron, let r denote the radius of its inscribed sphere. For $1 \leq k \leq 4$, let h_k denote the distance from the kth vertex to the plane of the opposite face. Prove that

$$\sum_{k=1}^4 \frac{h_k - r}{h_k + r} \geq \frac{12}{5}.$$

Problem 40. Let ABC be an equilateral triangle with center O and circumradius r. Given $R > r$, let ρ be a circle about O of radius R. All points named 'P' are on ρ.

(a) Prove that $PA^2 + PB^2 + PC^2 = 3(R^2 + r^2)$.

(b) Prove that $\min_{P\in\rho} PA\cdot PB\cdot PC = R^3 - r^3$ and that $\max_{P\in\rho} PA\cdot PB\cdot PC = R^3 + r^3$.

(c) Prove that the area of a triangle with side-lengths PA, PB and PC is $\frac{\sqrt{3}}{4}(R^2 - r^2)$.

(d) Prove that if H, K, and L are the respective projections of P onto $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$, then the area of triangle HKL is $\frac{3\sqrt{3}}{\boxed{116}}(R^2 - r^2)$.

(e) With the same notation, prove that $HK^2 + KL^2 + HL^2 = \frac{9}{4}(R^2 + r^2)$.

Problem 41. Let x_1, x_2, x_3,... be a sequence of positive numbers such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \frac{\ln x_n}{x_1+x_2+x_3+\cdots+x_n}$ is a negative number. Prove that $\lim_{n\to\infty} \frac{\ln x_n}{\ln n} = -1$.

Problem 42. Let x, y, and z be positive numbers such that $x + y + z = 3$. Prove that

$$\frac{x^4+x^2+1}{x^2+x+1} + \frac{y^4+y^2+1}{y^2+y+1} + \frac{z^4+z^2+1}{z^2+z+1} \geq 3xyz \tag{10}$$

Problem 43. Let ABC be an acute triangle, and let B_1 and C_1 be the points where the altitudes from B and C intersect the circumcircle. Let X be a point on arc $\overset{\frown}{BC}$, and let B_2 and C_2 denote the intersections of XB_1 with $\overline{AC}$ and XC_1 with $\overline{AB}$. Prove that the line $\overline{B_2C_2}$ contains the orthocenter of ABC.

Problem 44. For $n \in \mathbb{N}$, $k \in \mathbb{R}$, $k > n \geq 3$, we let x be the unique solution $(x > 1)$ of the equation

$$1 + x + x^2 + \cdots + x^{n-1} = k. \tag{*}$$

Show that the following estimates of x take place

$$\delta_2 + \frac{1}{n} < x < \delta_2 \left(1 + \frac{2k\delta_2^2}{(n-2)(n-2+\delta_1)+2\delta_2}\right), \tag{**}$$

where $\delta_1 = \sqrt{n^2+4k-4n}$ and $\delta_2 = 1 - \frac{1}{k}$.

Problem 45. Given a convex quadrilateral ABCD with O the intersection of its diagonals, let us denote by $[\alpha, \beta, \gamma, \delta]$ the ordered quadruple of positive numbers such that

$$\frac{[ABCD]}{[ODC]} = \alpha, \frac{[ABCD]}{[OCB]} = \beta, \frac{[ABCD]}{[OBA]} = \gamma, \text{ and } \frac{[ABCD]}{[OAD]} = \delta,$$

where $[XYZ...]$ means the area of the polygon $XYZ...$

(a) Show that the only quadruples of natural numbers possible (up to cyclic permutation and orientation) are $[2, 4, 12, 6]$, $[2, 3, 15, 10]$, $[3, 6, 6, 3]$ and $[4, 4, 4, 4]$.

(b) For the quadruple $[2, 4, 12, 6]$, show that if in addition $m(\angle DMC) = m(\angle DBC) = 90°$, then the quadrilateral is unique up to similarity and $\triangle MNE$ is congruent to $\triangle DNO$, where M, E, N and F are the midpoints of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$ respectively.

Problem 46. Let ℓ be the value of the side lengths of the largest equilateral triangle inscribed in $[0, 1]^4$ (with the Euclidean distance). Show that

$$2\sqrt{2-\sqrt{2}} \leq \ell \leq \sqrt{8/3}.$$

Problem 47. Show that for every $n \in \mathbb{N}$, $n \geq 2$, we have

$$\sum_{k=1}^{n}(1 - \frac{2k-1}{n}) \cot \frac{(2k-1)\pi}{2n} = \sum_{k=1}^{n-1} \csc \frac{k\pi}{n}, \tag{11}$$

Problem 48. For any commutative ring R with identity, let $R^\star$ denote the group of units of R and $R[X]$ the ring of polynomials in x over R. If, for each integer $m > 1$, $\mathbb{Z}_m$ denotes the ring of integers modulo m, for which m is $\mathbb{Z}_m[X]^\star = \mathbb{Z}_m^\star$?

Problem 49. Let a, b, and c be the lengths of the sides and s the semi-perimeter of the triangle ABC. Prove that

$$(a+b-c)^{a+b+s}(a-b+c)^{b+c+s}(-a+b+c)^{c+a+s} \leq a^{\frac{a}{2}+2s}b^{\frac{b}{2}+2s}c^{\frac{c}{2}+2s}. \tag{12}$$

Problem 50. Assume that a_1, a_2, ..., a_n $(n \geq 3)$ are positive numbers such that

$$a_1 + a_2 + \cdots + a_n \geq a_1 a_2 \cdots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

Prove that

$$a_1^n + a_2^n + ... + a_n^n \geq n(a_1^{n-1}a_2^{n-1} \cdots a_n^{n-1}). \tag{13}$$

Find a necessary and sufficient condition for equality to hold.

Problem 51. Let a, b, c, and d be real $n \times 1$ vectors and A and B be $n \times n$ real positives definite matrices. Prove that

$$2(a'c)(b'd) \leq (a'A^{-1}a)(b'Bb) + (c'Ac)(d'B^{-1}d). \qquad (14)$$

Problem 52. Evaluate

$$I := \int_0^\infty \int_0^\infty \frac{\ln^2(1+x^2+y^2)}{(1+x^2)(1+y^2)} dxdy.$$

Problem 53. Find all rings R (not assumed to be commutative or to contain an identity) with the following two properties:

i) not every element of R is nilpotent, and

ii) $x^2 = y^2$, for any nonzero $x, y \in R$.

Problem 54. Let a, b, c be positive reals such that $a+b+c=3$. Show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 1 + 2\sqrt{\frac{a^2+b^2+c^2}{3abc}}.$$

Problem 55. Suppose f is a function defined on an open interval I such that $f''(x) \geq 0$ for all $x \in I$, and $[a,b] \subset I$. Prove that

$$\int_0^1 f(a+(b-a)y)dy \geq \int_0^1 f(\frac{3a+b}{4} + \frac{b-a}{2}y)dy.$$

Problem 56. Let a, b, $c > 0$ and $a+b+c=3$. Prove that

$$\sum_{cycl} \frac{bc}{\sqrt[4]{a^2+3}} \leq \frac{3\sqrt{2}}{2} \qquad (15)$$

where the sum is over all cyclic permutation of (a,b,c) and equality occurs when $a=b=c=1$.

Problem 57. Let P be a point in a plane of $\triangle ABC$ which is not on the sidelines AB, AC, or BC and let cevians AA', BB', CC' of triangle ABC intersect at P. Show that triangles $B'A'C$ and $A'C'B$ have the same orientation and equal areas if and only if A' is a midpoint of $\overline{BC}$.

Problem 58. Let $C(\mathbb{R})$ be the ring (under pointwise addition and multiplication) of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Suppose that S is a subring (not necessarily with 1) of $C(\mathbb{R})$ which contains only monotone functions (the monotonicity need not be strict). Must every member of S be a constant function? Prove or provide a counterexample.

Problem 59. Let ABC be a triangle inscribed in a circle and circumscribed to another circle of center I. If $\overline{AM}$ is the median which cuts the interior circle into the points K and L, and the exterior circle in the point P, such that AK, KL, and LP are congruent, then prove that

$$\frac{2}{a} = \frac{1}{b} + \frac{1}{c},$$

where a, b, and c are the sides of the triangle.

Problem 60. Let X be a set and $f : X \to X$ be a function. A subset Y of X is said to be closed under f provided that whenever $y \in Y$, $f(y) \in Y$. Prove or disprove: There exists an uncountable set X and a function $f : X \to X$ with the following property:

(*) for any subsets Y and Z of X which are closed under f, either $Y \subset Z$ or $Z \subset Y$.

Problem 61. Let A, B, C be three independently selected, uniformly distributed points on the unit sphere S_3 in $\mathbb{R}^4$. With probability one, the points A, B, C determine a unique chordal triangle T inscribed within S_3, with sides as straight lines through the interior of S_3. What is the probability that T is acute?

Problem 62. Let $ABCD$ be a square with sides of length a. Suppose K and L are points on the sides $\overline{BC}$ and $\overline{CD}$, respectively, so that the perimeter of triangle KCL is $2a$. If triangle AKL has minimum area, determine the measures of its angles.

Problem 63. Let I and G be the points of intersection of the bisectors and medians of a triangle ABC with sides a, b and c, respectively. Prove that

$$|IG|^2 = \frac{1}{9}\left\{2(b^2+c^2) - 3(b+c)(b+c-a) - a^2 + \frac{9bc(b+c-a)}{(a+b+c)}\right\}. \quad (16)$$

Problem 64. Let ABC be an isosceles triangle with $AB = AC$ and $\angle A = 100°$. Let D be a point on side $\overline{AB}$ so that $\angle BCD = 10°$ and let E be a point on side $\overline{BC}$ so that $EC = AC$. Find a point K on the line segment $\overline{CD}$ so that triangles KAD and KCE have equal area.

Problem 65. Let $a > 0$ be a real number and let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence defined by the recurrence relation

$$x_1 = 1; \text{ and } x_{n+1} = x_n + \frac{a}{x_1 + x_2 + \cdots + x_n}, \text{ for } n \geq 1. \quad (17)$$

(i) Prove that $\lim_{n\to\infty} x_n = \infty$.
(ii) Calculate $\lim_{n\to\infty} \frac{x_n}{\sqrt{\ln n}}$.

Problem 66. Let ABCD be a convex quadrilateral and O be the intersecting point of its diagonals. Let K, L be points on the side CD such that $\overline{OK}||\overline{AD}$ and $\overline{OL}||\overline{BC}$. Supposing that $DK^2 + LC^2 = KL^2$, determine the ratio of the areas ABCD and OCD.

Problem 67. Let f be a differentiable function such that, for some a, b satisfying $0 < a < b < 1$ we have

$$\frac{\int_0^a f(x)dx}{a(1-a)} + \frac{\int_b^1 f(x)dx}{b(1-b)} = 0. \quad (18)$$

If $K = \int_0^a f(x)dx$, prove that

$$\left| \int_0^1 f(x)dx \right| \leq \frac{b-a}{2} \sup_{x\in(0,1)} \left| f'(x) + \frac{2K}{a(1-a)} \right|. \tag{19}$$

Problem 68. We call *toroidal chess board* a regular chess board (of arbitrary dimensions) in which the opposite sides are identified in the same direction. Show that the maximum number of kings on a toroidal chess board of dimensions $m \times n$ $(m, n \in \mathbb{N})$ such that each king attacks no more than six other kings is less than or equal to $\frac{4mn}{5}$ and the inequality is sharp.

Problem 69. (a) Show that the probability of a point $P(x, y, z)$, chosen at random with uniform distribution in $[0, 1]^3$, to be at a distance to the origin of at most $\sqrt{2}$ is $\frac{(15-8\sqrt{2})\pi}{12}$.

(b) Prove that

$$\int_0^{\pi/4} \frac{\cos^{3/2} 2\theta}{\cos^3 \theta} d\theta = \frac{(4\sqrt{2}-5)\pi}{4}.$$

Problem 70. A stick is broken at random at two points (each point is uniformly distributed relative to the whole stick) and the parts' length are denoted by r, s, and t. Show that the probability of the existence of a triangle encompassing three circles of radii r, s and t each side tangent to two of the circles and the circles are mutually externally tangent, is equal to $\frac{5}{27}$.

Problem 71. (I) (Gauss) Show that a prime p can be written as a^2+8b^2 for some $a, b \in \mathbb{N}$ if and only if it can be written as $x^2 + 16y^2$ for some $x, y \in \mathbb{N}$.

(II) Show that a prime p can be written as $a^2 + 16b^2$ for some $a, b \in \mathbb{N}$ if and only if $p \equiv 1 \pmod 8$.

(III*) Show that a prime p can be written as $a^2 + 32b^2$ for some $a, b \in \mathbb{N}$ if and only if $p \equiv 1 \pmod 8$ and there exists a solution x of the equation

$$(x^2 - 1)^2 \equiv -1 (mod\ p).$$

Problem 72. For which positive integers n can the set $\{1, 2, ..., 2n\}$ be partitioned into n two element subsets so that the sum of the two numbers in each subset is a perfect square?

Problem 73. Let a, b, c, d be nonnegative real numbers with $a+b = c+d = 1$. Determine the maximum value of

$$(a^2 + b^2)(a^2 + d^2)(b^2 + c^2)(b^2 + d^2)$$

and determine conditions under which the maximum is attained.

Problem 74. Let P and Q be idempotent, hermitian matrices of the same dimension and rank. Prove that $PQP = P$, then $P = Q$.

Problem 75. Let $\triangle ABC$ be a triangle with circumcenter O, perimeter P, area K. Prove that if

$$\frac{BC}{P} = \frac{1}{3} = \frac{[OBC]}{K}$$

then $\triangle ABC$ is equilateral. (Here [XYZ] denotes the area of triangle XYZ.)

Problem 76. Let N be a positive integer. Prove that there is a positive integer n such that $n^2 + 3$ is divisible by at least N distinct primes.

Problem 77. Let A be an $n \times n$ matrix with complex entries such that $A^2 = A^*$, where A^* denotes the conjugate transpose of A. Show that

(a) rank$(A + A^*)$=rank (A)

(b) $I_n + A$ is nonsingular.

Problem 78. Let x_1, x_2, ..., $x_n \geq e$. Prove that

$$x_1^{\frac{x_1+x_2+\cdots+x_n}{x_1}} + x_2^{\frac{x_2+x_3+\cdots+x_n}{x_2}} + \cdots + x_{n-1}^{\frac{x_{n-1}+x_n}{x_{n-1}}} + x_n \geq x_1 + 2x_2 + \cdots + (n-1)x_{n-1} + nx_n.$$

Problem 79. Find a function $f : [0,1] \to [0,1]$ such that for each nontrivial interval $I \subset [0,1]$, we have $f(I) = [0,1]$.

Problem 80. A point is selected at random from the region inside of a regular n-gon. What is the probability that the point is closer to the center of the n-gon that it is to the n-gon itself?

Problem 81. Let a, b and c be nonnegative real numbers. Find the value of

$$L := \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt{n^2+kn+a}}{\sqrt{n^2+kn+b}\sqrt{n^2+kn+c}}.$$

Problem 82. Let x, y, and z be positive real numbers with $x+y+z = xyz$. Find the minimum value of

$$\sqrt{1+x^2} + \sqrt{1+y^2} + \sqrt{1+z^2},$$

and find all (x, y, z) for which the minimum occurs.

Problem 83. Let ABC be a triangle, let E be a fixed point on the interior of side $\overline{AC}$, and let F be a fixed point on the interior of side $\overline{AB}$. For P on $\overline{EF}$, define

$$\rho(P) = \frac{[PBC]^2}{[PCA][PAB]}.$$

For which P does $\rho(P)$ take on its minimum value? What is this minimal value?

Problem 84. Let ABCD be a convex quadrilateral, let X and Y be the midpoints of the sides $\overline{BC}$ and $\overline{DA}$ respectively, and let O be the point of intersection of diagonals of $ABCD$. Prove that O lies inside of the quadrilateral $ABXY$ if and only if

$$Area(AOB) < Area(COD).$$

Problem 85. Let u and v be positive real numbers. Prove that

$$\frac{1}{8}\left(17-\frac{2uv}{u^2+v^2}\right) \leq \sqrt[3]{\frac{u}{v}}+\sqrt[3]{\frac{v}{u}} \leq \sqrt{(u+v)(\frac{1}{u}+\frac{1}{v})}$$

Find conditions under which equality holds.

Problem 86. Let f be a continuous real-valued function defined on $[0,1]$ and satisfying

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx.$$

Prove that there exists a real number c, $0 < c < 1$, such that

$$cf(c) = \int_0^c xf(x)dx.$$

Problem 87. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with a continuous derivative such that $f(0) = f(1) = -\frac{1}{6}$. Prove that

$$\int_0^1 (f'(x))^2 dx \geq 2\int_0^1 f(x)dx + \frac{1}{4}. \tag{20}$$

Problem 88. Prove that (quicky)

$$\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{1}{(1+t^2)^2}dt = \frac{\pi}{12}.$$

Problem 89. Prove that there exists a real number α satisfying $\lfloor \alpha^n \rfloor \equiv n \pmod 5$ for all $n \in \mathbb{N}$.

Problem 90. Find the area of the region bounded by the curve $x^{2n+1} + y^{2n+1} = (xy)^n$, $n = 1, 2, 3....$

Problem 91. Numbers r, s, and t are chosen independently and uniformly at random from the interval $(0, 1]$. Circles with radii r, s, and t are then constructed so that they are pairwise externally tangent. What is the probability that the circles can be enclosed in a triangle, each of whose sides is tangent to two of the circles?

Problem 92. Given a pyramid ABCD with bottom triangular face ABC, having $BC = a$, $AC = b$, and $AB = c$, we denote the dihedral angles made by the lateral faces BCD, ACD, ABD with the bottom by α, β, and γ, respectively. If these angles are acute, find the radius r of the sphere inscribed in the pyramid (tangent to all faces).

Problem 93. Let I and G be the points of intersection of the bisectors and medians of a triangle ABC with sides a, b and c, respectively. Prove that

$$|IG|^2 = \frac{1}{9}\left\{2(b^2+c^2) - 3(b+c)(b+c-a) - a^2 + \frac{9bc(b+c-a)}{(a+b+c)}\right\}. \quad (21)$$

Problem 94. Prove that if x, y, $z > 0$, then

$$\frac{(x+y+z)^3 - 2(x+y+z)(x^2+y^2+z^2)}{xyz} \leq 9.$$

Problem 95. For each nonnegative integer n, let

$$x_n = \frac{1}{(b-a)^{2n+1}} \int_a^b (x-a)^n (b-x)^n dx$$

, $a < b$. Show that $\sum_{n=1}^{\infty} x_n$ is convergent.

Problem 96. For every positive integer k, consider the series

$$\begin{aligned} S_k = \ & \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{k}\right) - \left(\tfrac{1}{k+1} + \tfrac{1}{k+1} + \cdots + \tfrac{1}{2k}\right) \\ & + \left(\tfrac{1}{2k+1} + \tfrac{1}{2k+2} + \cdots + \tfrac{1}{3k}\right) - \left(\tfrac{1}{3k+1} + \tfrac{1}{3k+2} + \cdots + \tfrac{1}{4k}\right) + \cdots \end{aligned}$$

(a) If $H_k = \sum_{m=1}^{k} \frac{1}{m}$ is the harmonic series, show that there exists an $\alpha \in (0,1)$ such that for every k,

$$\alpha H_k < S_k < H_k. \tag{22}$$

(b) Prove that

$$\lim_{k\to\infty} H_k - S_k = \ln \frac{\pi}{2}. \tag{23}$$

(c) Find a closed form of S_k (with a finite number of summands). Check your solution by evaluating

$$\lim_{k\to\infty} S_8 = \pi \left(\frac{1}{16} + \frac{\sqrt{2}}{8} + \frac{1}{4}\sqrt{2+\sqrt{2}} \right) + \frac{1}{8} \ln 2. \tag{24}$$

(d) Given that γ is the Euler-Mascheroni constant, prove that

$$\lim_{k\to\infty} \left(\frac{1}{2k} \sum_{m=1}^{k-1} \frac{\pi}{\sin \frac{m\pi}{k}} \right) - \ln \frac{2k}{\pi} = \gamma. \tag{25}$$

Problem 97. In this problem all variables represent positive integers. (i) Prove that $a^2 + b^2 = ab$ is impossible. (ii) If a, b, c satisfy $a^2 + b^2 + c^2 = abc$, then prove that $27|abc$. (iii) If a, b, c, d satisfy $a^2 + b^2 + c^2 + d^2 = abcd$, then prove that $16|abcd$. (iv$\star$) Prove or disprove: if a, b, c, d, e satisfy $a^2 + b^2 + c^2 + d^2 + e^2 = abcde$, then $9|abcde$. More generally, what might be true for the sum of n squares?

Problem 98. Let f and f^2 be Riemann-integrable functions defined on $[0,1]$, and let g be a twice-differentiable function defined on $[0,1]$ such that $g(0) = 1$.

(a) Show that

$$\lim_{n\to\infty} \prod_{k=1}^{n} g(\frac{1}{n} f(\frac{k}{n})) = exp(g'(0) \int_0^1 f(x)dx)$$

(b) Find a suitable choice of the functions f and g to solve Problem 1892 from Mathematics Magazine (proposed by Jose Luis Diaz-Barrero):

$$\lim_{n\to\infty} \frac{1}{n^n} \prod_{k=1}^{n} \frac{n\sqrt{n} + (n+1)\sqrt{k}}{\sqrt{n} + \sqrt{k}} = \frac{4}{e}.$$

Problem 99. Given x_1, x_2, x_3, x_4, x_5, $x_6 \in (0, \infty)$ such that

$$\frac{1}{x_1+x_2}+\frac{1}{x_3+x_4}+\frac{1}{x_5+x_6}=1$$

prove that

$$\left(\sum_{i=1}^{6} x_i\right)^2 \left(9+\sum_{i=1}^{6} x_i\right) \geq 54(x_1+x_2)(x_3+x_4)(x_5+x_6).$$

Problem 100. Consider a set of five distinct positive real numbers such that if we take all products of pairs of these numbers then only seven distinct numbers are formed. Thus if the numbers are $0 < x_1 < x_2 < x_3 < x_4 < x_5$, if we look at the set formed from all products $x_i x_j$ with $i \neq j$ then there are only seven distinct numbers. Prove the x_i's form an geometric progression; in other words, there is an r such that $x_{i+1} = r x_i$ for $i \in \{1, 2, 3, 4\}$.

Problem 101. A term a_k of a sequence $\{a_n\}$ is called a local extreme if either $a_{k-1} \leq a_k \geq a_{k+1}$ or $a_{k-1} \leq a_k \geq a_{k+1}$.

(a) If a sequence has infinitely many local extreme terms prove that the sequence is convergent if and only if the subsequence of all local extreme terms is convergent.

(b) Show that Part (a) is no longer true if in the definition of a local extreme $\leq$ and $\geq$ are replaced by $<$ and $>$ respectively.

Problem 102. A stick is broken into three pieces at random (the two breaking points are, simultaneously, chosen at random with uniform distribution). Show that the probability that the three segments are the heights of a triangle is equal to

$$\frac{4}{25}\left(3\sqrt{5}\ln\frac{3+\sqrt{5}}{2}-5\right).$$

Problem 103. Let $\Omega = \{0, 1, 2, 3, ...\}$, r a real number such that $0 < r < 1$, and the discrete probability on Ω defined by $P_r(E) := \frac{1-r}{r}\sum_{k\in E} r^k$ if $0 \notin E$ and $P_r(0) = 0$. Show that there are uncountable many triple of events (A, B, C) which are mutually independent, (i.e. $P(A\cap B) = P(A)P(B)$, $P(A\cap C) = P(A)P(C)$, $P(C\cap B) = P(C)P(B)$ and $P(A\cap B\cap C) = P(A)P(B)P(C)$).

Problem 104. Let $\alpha \in (0,1)$ be an arbitrary irrational number. Construct a sequence $\{a_n\}$ of real numbers with the following two properties:

$$(a) \quad \{a_n | n \in \mathbb{N}\} \subset \{\frac{1}{n} | n \in \mathbb{N}\} \cup \{\frac{n-1}{n} | n \in \mathbb{N}\},$$

$$(b) \qquad \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \alpha.$$

Problem 105. Let G be the region in the three-dimensional space located in the plane $x = 0$ and bounded by $|z| = 1 - y^2$ with $-1 \le y \le 1$. Denote by S_1 the solid generated by rotating G about the z-axis and S_2 be the solid generated by rotating G about the y-axis. Show that the volume of $S_1 \cap S_2$ is $\frac{1}{30}(43\sqrt{2} - 32)\pi$ and the volume of $S_1 \cup S_2$ is $\frac{1}{30}(94 - 43\sqrt{2})\pi$.

Problem 106. Consider an acute triangle $\triangle ABC$ and let O be its circumcenter. Denote the distances of O to the sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, by x, y and z respectively.

(a) Show that the radius R, of the circle circumscribed to the triangle $\triangle ABC$, must satisfy the equation

$$R^3 - (x^2 + y^2 + z^2)R - 2xyz = 0. \tag{26}$$

(b) Given three positive real numbers x, y and z, there exists one and only one acute triangle with the distances of the circumcenter to the sides exactly equal to x, y and z. Show that the previous statement is true if one changes the adjective acute with obtuse.

(c) Show that (26) has infinitely many integer solutions in $(x, y, z, R) \in \mathbb{N}^4$ in which x, y and z are different.

(d^*) What are all the integer solutions of (26)?

Problem 107. We let $I = \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)^3}$ and $J = \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)^3}$.

(a) Show that

$$I + J = \frac{5\pi^3\sqrt{3}}{243} \quad \text{and} \tag{27}$$

$$I - J = \frac{13}{18}\zeta(3). \tag{28}$$

(b) Prove that

$$\zeta(3) = \frac{9}{13}\int_0^1 \frac{(\ln x)^2}{x^3+1}dx - \frac{18}{13}J. \tag{29}$$

Here, ζ denotes the Riemann zeta function.

Problem 108. Prove that

$$10|x^3 + y^3 + z^3 - 1| \leq 9|x^5 + y^5 + z^5 - 1| \tag{30}$$

for real numbers x, y and z with $x + y + z = 1$. When does equality hold?

Problem 109. Let

$$a_1(k,n) = \frac{1}{8}[9^k(24n+5) - 5],\ a_2(k,n) = \frac{1}{8}[9^k(24n+13) - 5],$$

$$a_3(k,n) = \frac{1}{8}[3 \cdot 9^k(24n+7) - 5],\ \text{and}\ a_4(k,n) = \frac{1}{8}[3 \cdot 9^k(24n+23) - 5].$$

Show that for each nonnegative integer m there is a unique integer triple (j,k,n) with $j \in \{1,2,3,4\}$ and $k,n \geq 0$ such that $m = a_j(k,n)$.

Problem 110.