Easy Putnam Problems [1]

1. *Prove that every nonzero coefficient of the Taylor series of*

$$(1-x+x^2)e^x$$

about $x=0$ us a rational number whose numerator (in lowest terms) is either 1 or a prime number.

2. *For a natural number n we define $R(n)=n-m^2$, where m is the biggest natural number such that $m^2 \le n$. For $M \in \mathbb{N}$, define the recurrent sequence $x_1 = M$, and $x_{k+1} = x_k + R(x_k)$ for $k \ge 1$. For what values of M, $\{x_k\}_k$ becomes eventually constant?*

3. *Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them. (See Figure 1)*

Figure 1

4. *A composite (positive integer) is a product ab with a and b not necessarily distinct integers in 2, 3, 4, ... Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z, positive integers.*

5. *Given a positive integer n, what is the largest k such that the numbers 1, 2, 3,...,n can be put into k boxes so that the sum of the numbers in each box is the same? [When n = 8, the example $\{1,2,3,6\}$, $\{4,8\}$, and $\{5,7\}$ shows that the largest k is at least 3.]*

6. *Let f be a real-valued function on the plane such that for every square ABCD in the plane, f(A) + f(B) + f(C) + f(D) = 0. Does it follow that f(P) = 0 for all points P in the plane?*

7. *Find polynomials f, g and h, if they exist, such that for all x*

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$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x+2 & \text{if } -1 \le x \le 0 \\ -2x+2 & \text{if } x > 0. \end{cases}$$

8. *A right circular cone has a base of radius 1 and a height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?*

9. *Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.*

10. *Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any three (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.*

11. *Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.*

12. *Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)*

13. *A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC? (See Figure 2)*

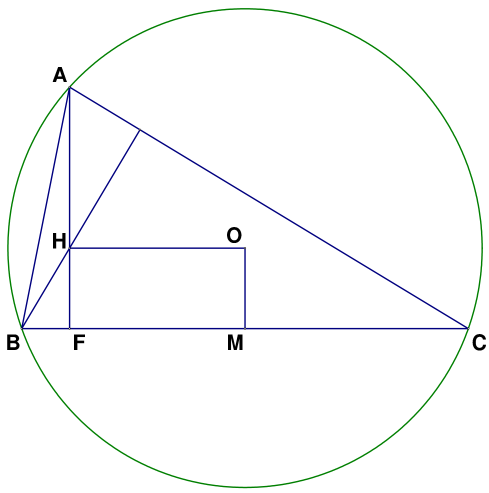

Figure 2

14. *Consider a set* S *and a binary operation* $\diamond$, *i.e., for every* $a, b \in S$, $a \diamond b \in S$. *Assume* $(a \diamond b) \diamond a = b$ *for all* $a, b \in S$. *Prove that* $a \diamond (b \diamond a) = b$ *for all* $a, b \in S$.

15. *Let* n *be a fixed positive integer. How many ways are there to write* n *as a sum of positive integers,* $n = a_1 + a_2 + \ + a_k$, *with* k *an arbitrary positive integer and* $a_1 \leq a_2 \leq \ \leq a_k \leq a_1 + 1$? *For example, with* $n = 4$ *there are four ways:* 4, $2+2$, $1+1+2$, $1+1+1+1$.

16. *Let* f *be a three times differentiable function on* $\mathbb{R}$ *having at least five distinct real zeroes. Show that*

$$f + 6f' + 12f'' + 8f'''$$

has at least two distinct real zeroes.

17. *Find the minimum value of*

$$\frac{(x+\frac{1}{x})^6 - (x^6 + \frac{1}{x^6}) - 2}{(x+\frac{1}{x})^3 + (x^3 + \frac{1}{x^3})}$$

for $x > 0$.

18. *Right triangle* ABC *has a right angle at* C *and* $\angle BAC = t$; *the pint* D *is chosen on* AB *so that* $|AC| = |AD| = 1$; *the point* E *is chosen on* BC *so that* $\angle CDE = t$. *The perpendicular on* $\overline{BC}$ *at* E *meets* AB *at* F. *Evaluate*

$$\lim_{t \to 0} |EF|.$$

19. *Show that the curve* $x^3 + 3xy + y^3 = 1$ *contains only one set of three distinct points, which are vertices of an equilateral triangle, and find its area.*

20. *Let* f *be a non-constant polynomial with positive integer coefficients. Prove that if* n *is a positive integer, then* $f(n)$ *divides* $f(f(n)+1)$ *if and only if* $n = 1$.

21. *Define an* <u>*id*</u> *set to be a set which has its own cardinality (number of elements) as an element. Find the number of subsets of* $[m] := \{1, 2, ..., m\}$ *which are minimal id sets, i.e., sets which are id sets but none of their proper subsets are id sets.*

22. *Find a nonzero polynomial* $P(x, y)$ *such that* $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ *for all real numbers* a. *(Note:* $\lfloor x \rfloor$ *is the greatest integer less than or equal to* x.*)*

23. *Inscribe a rectangle of base* b *and height* h *and an isosceles triangle of base* b *in a circle of radius one as shown. For what value of* h *do the rectangle and triangle have the same area?*

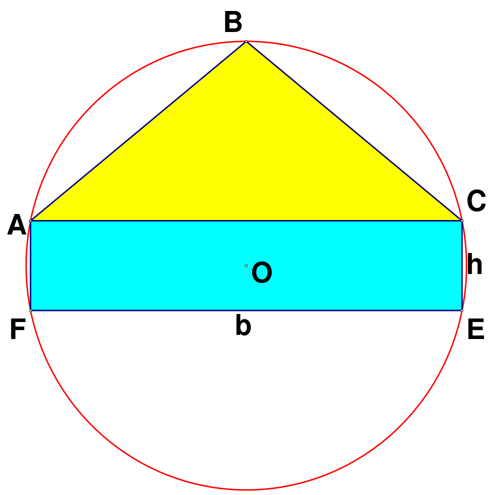

Figure 2

24. Let A be the $n \times n$ matrix whose entry in the $i-th$ row and $j-th$ column is

$$\frac{1}{min(i,j)}$$

for $1 \leq i,j \leq n$. Compute $det(A)$.

25. Find the volume of the region of points (x,y,z) such that

$$(x^2+y^2+z^2+8)^2 \leq 36(x^2+y^2).$$

26. Let S be a class of functions from $[0,1)$ to $[0,1)$ that satisfies:

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x)+g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x)-g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

27. What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A rational point is a point both of whose coordinates are rational numbers.)

28. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

29. For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

30. *Let d_1, d_2, ..., d_{12} be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, and k such that d_i, d_j, and d_k are the side lengths of an acute triangle.*

31. *Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,*

$$\frac{10}{9} = \frac{2!5!}{3!(3!)(3!)}.$$