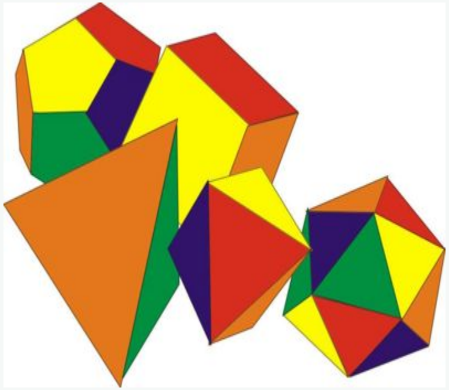
Existence of regular polyhedrons in space with integer coordinates



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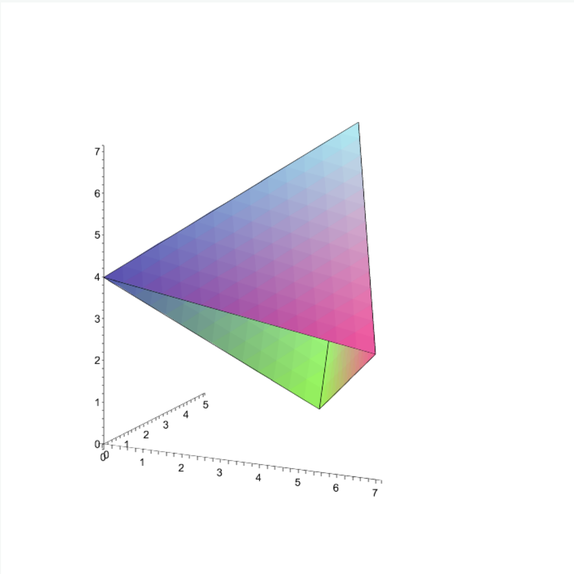
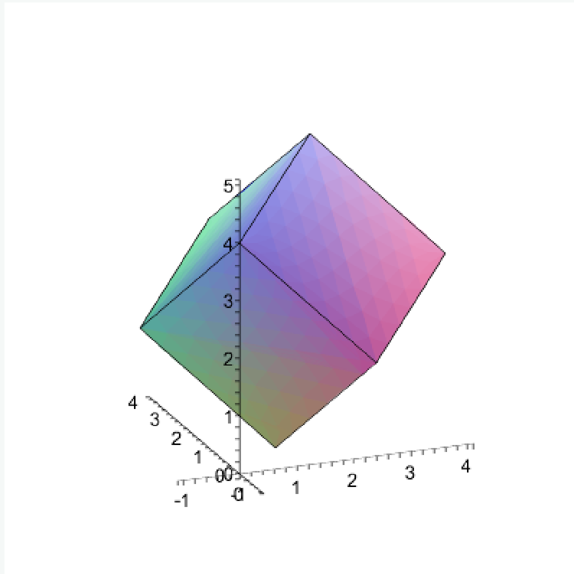
Two examples



$Figure\ 1a$, $\{(0,0,4),(7,0,3),(3,5,0),(4,5,7)\}$



$Figure\ 1b$,
$\{(1,4,3),(3,3,1),(1,1,0),(-1,2,2),$
$(2,2,5),(4,1,3),(2,-1,2),(0,0,4)\}$

Euclidean Distance: $d(A,B) := \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2 + (a_z - b_z)^2}$, where $A = (a_x, a_y, a_z)$ and $B = (b_x, b_y, b_z)$

An example of a regular octahedron in $\mathbb{Z}^3$:

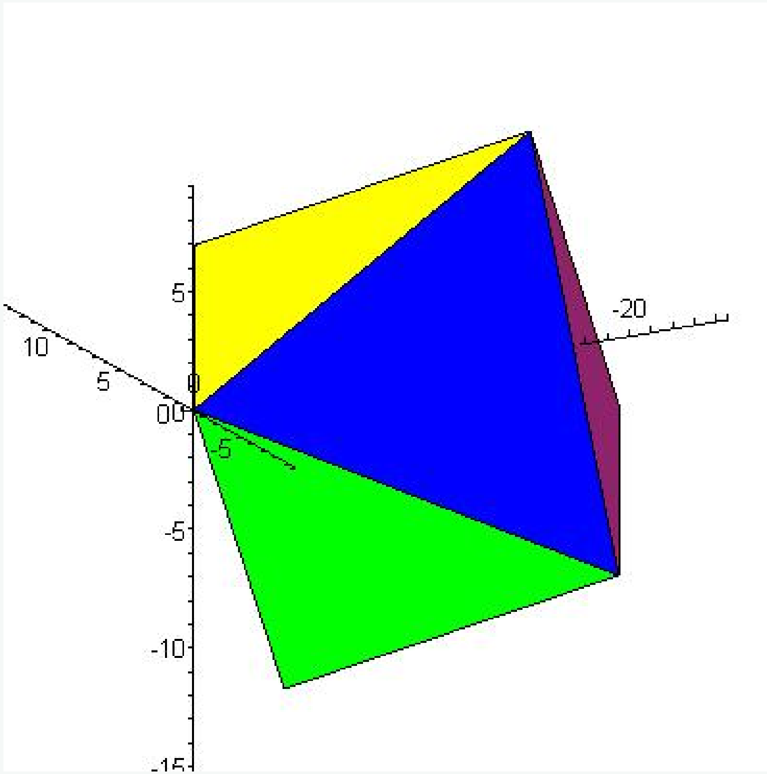


Figure 2, $\{(0,0,0),(1,-16,9),(-8,-15,-7),(16,-9,1),(7,-8,-15),(8,-24,-6)\}$

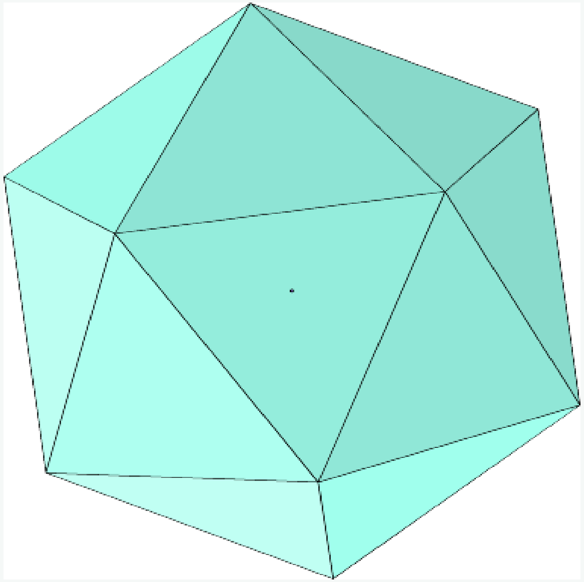


Figure 3 (a): Regular Icosahedron

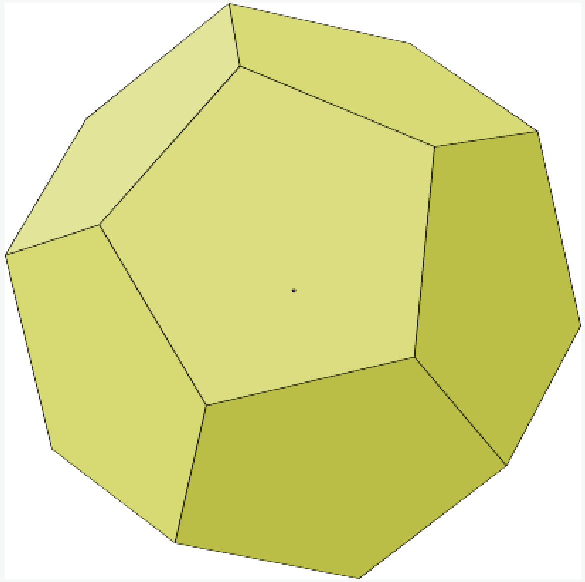


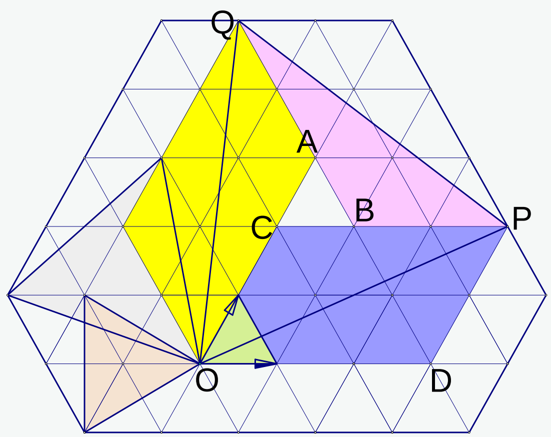
Figure 3 (b): Regular Dodecahedron

Theorem A(JNT 2011, A. Markov) *There is no regular icosahedron or regular dodecahedron in* $\mathbb{Z}^3$.

The lattice $\mathcal{P}_{a,b,c}$ containing equilateral triangles

$$\mathcal{P}_{a,b,c} := \{(\alpha, \beta, \gamma) \in \mathbb{Z}^3 | \quad a\alpha + b\beta + c\gamma = e, \qquad (1)$$

$$\boxed{a^2 + b^2 + c^2 = 3d^2}, \quad a, b, c, d, e \in \mathbb{Z}\}.$$



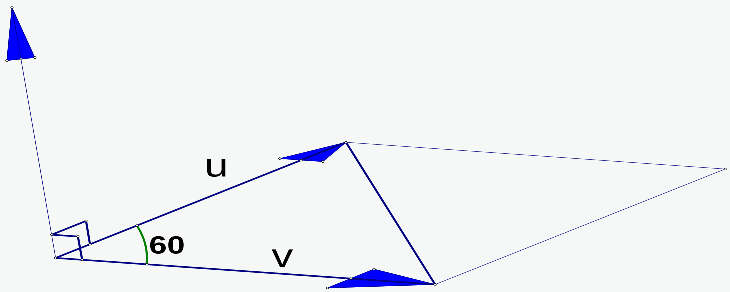


Figure 4

$$u \times v = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}, \qquad a, b, c \in \mathbb{Z}$$

$$|u \times v| = 2\frac{|u| \cdot |v| \sin(60°)}{2} = \sqrt{3}\frac{\ell^2}{2} \Rightarrow$$

$$|u \times v|^2 = a^2 + b^2 + c^2 = 3(\frac{\ell^2}{2})^2 \Rightarrow$$

ℓ^2 must be divisibe by 2 and if $\ell^2/2 = d$ we have

$$a^2 + b^2 + c^2 = 3d^2.$$

Fact 1: *The cosine of the dihedral angle between two adjacent faces of a regular icosahedron is* $-\frac{\sqrt{5}}{3}$. *(Exercise)*

Fact 2: *The cosine of the dihedral angle between two planes* $\mathcal{P}_{a,b,c}$ *and* $\mathcal{P}_{a',b',c'}$ *is*

$$\cos\alpha = \frac{\overrightarrow{n_1} \cdot \overrightarrow{n_2}}{\|\overrightarrow{n_1}\| \|\overrightarrow{n_2}\|} = \frac{aa' + bb' + cc'}{3dd'}$$

Observation: *For a regular tetrahedron the cosine of the dihedral angle is* $\frac{1}{3}$, *for the cube is 0, and for regular octahedron is* $\frac{-1}{3}$.

Examples of planes $\mathcal{P}_{a,b,c}$

$d = 1$, [1, 1, 1]

$d = 3$, [1, 1, 5]

$d = 5$, [1, 5, 7]

$d = 7$, [1, 5, 11]

$d = 9$, [1, 11, 11] [5, 7, 13]

$d = 11$, [1, 1, 19], [5, 7, 17], [5, 13, 13]

$d = 13$, [5, 11, 19], [7, 13, 17]

$d = 15$, [1, 7, 25], [5, 11, 23], [5, 17, 19]

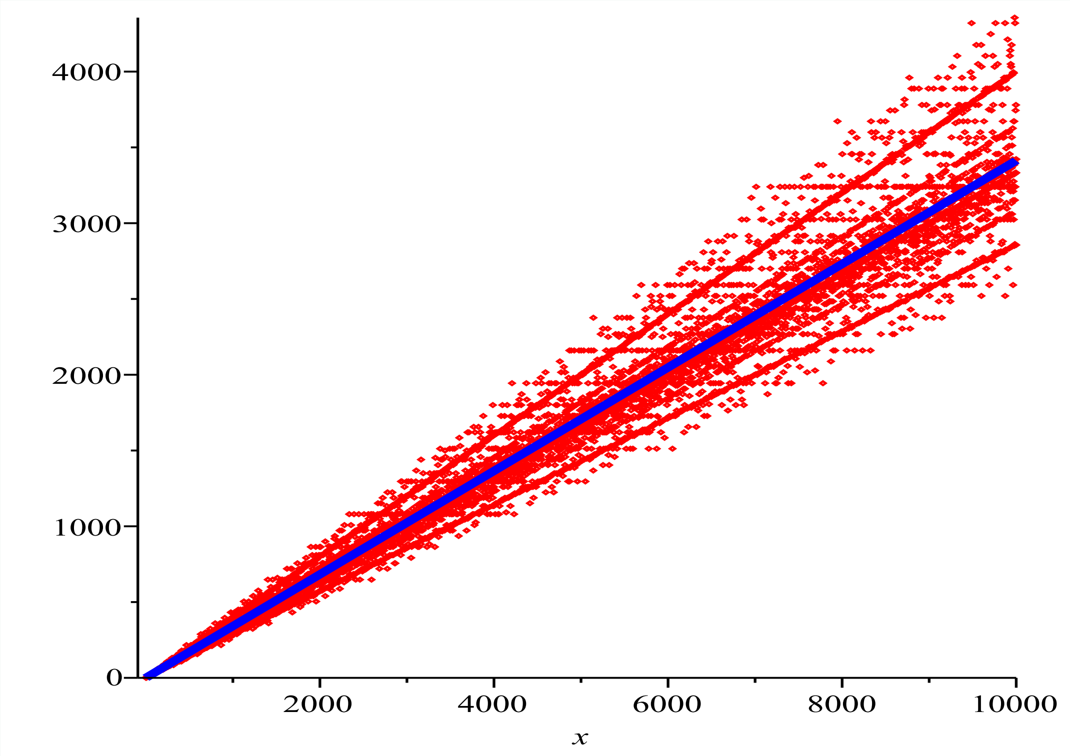
$d = 17$, [1, 5, 29], [7, 17, 23], [11, 11, 25], [13, 13, 23]

$d = 19$, [1, 11, 31], [5, 23, 23], [11, 11, 29], [13, 17, 25]

$d = 21$, [1, 19, 31], [11, 19, 29], [13, 23, 25]

Question: *What is the number of primitive solutions (a, b, c), given an odd positive integer d ?* We will denote this number by $\pi\epsilon(d)$.

Graph of $k \to \pi\epsilon(2k+1), k = 1, 2, ..., 10000$



Slope of the average line $m \approx 0.34131$ computed for the first 10 million odd numbers.

$$\boxed{\pi\epsilon(d) = \frac{\Lambda(d) + 24\Gamma_2(3d^2)}{48}} \tag{2}$$

$$\Lambda(d) := 8d \prod_{p|d, p \ prime} \left(1 - \frac{(\frac{-3}{p})}{p}\right), \tag{3}$$

Legendre symbol

$$(\frac{-3}{p}) = \begin{cases} 0 \ \text{ if } p = 3 \\ \\ 1 \ \text{ if } p \equiv 1 \text{ or } 7 \pmod{12} \\ \\ -1 \ \text{ if } p \equiv 5 \text{ or } 11 \pmod{12} \end{cases} , \tag{4}$$

$$\Gamma_2(d) = \begin{cases} 0 \ \text{ if d is divisible by a prime factor of the form 8s+5 or 8s+7, } \ s \geq 0, \\ \\ 2^{k-1} \quad \begin{cases} \text{where k is the number of distinct prime factors of d} \\ \\ \text{of d of the form 8s+1, or 8s+3 } (s \geq 0). \end{cases} \end{cases} \tag{5}$$

Theorem B : *The sub-lattice* $\mathcal{P}^{eq}{}_{a,b,c}$ *is generated by two vectors* $\overrightarrow{\zeta}$ *and* $\overrightarrow{\eta}$ *in the following sense:* $\mathcal{T}^{m,n}_{a,b,c} := \triangle OPQ$ *with* P, Q *in* $\mathcal{P}_{a,b,c}$, *is equilateral if and only if for some integers* m, n

$$\overrightarrow{OP} = m\overrightarrow{\zeta} - n\overrightarrow{\eta}, \quad \overrightarrow{OQ} = n\overrightarrow{\zeta} + (m-n)\overrightarrow{\eta}, \text{ with } \overrightarrow{\zeta} = (\zeta_1, \zeta_1, \zeta_2), \overrightarrow{\varsigma} = (\varsigma_1, \varsigma_2, \varsigma_3), \overrightarrow{\eta} = \frac{\overrightarrow{\zeta} + \overrightarrow{\varsigma}}{2}, \tag{6}$$

$$\begin{cases} \zeta_1 = -\dfrac{rac + dbs}{q} \\ \\ \zeta_2 = \dfrac{das - bcr}{q} \\ \\ \zeta_3 = r \end{cases}, \quad \begin{cases} \varsigma_1 = \dfrac{3dbr - acs}{q} \\ \\ \varsigma_2 = -\dfrac{3dar + bcs}{q} \\ \\ \varsigma_3 = s \end{cases}, \tag{7}$$

where $q = a^2 + b^2$ *and* r, s *can be chosen so that all six numbers in (3) are odd integers. The sides-lengths of* $\triangle OPQ$ *are equal to* $d\sqrt{2(m^2 - mn + n^2)}$.

Moreover, r *and* s *can be constructed in such a way that the following properties are also verified:*

(i) r *and* s *satisfy* $2q = s^2 + 3r^2$ *and as a result* $2(b^2 + c^2) = \varsigma_1^2 + 3\zeta_1^2$ *and* $2(a^2 + c^2) = \varsigma_2^2 + 3\zeta_2^2$

(ii) $r = r'\omega\chi$, $s = s'\omega\chi$ *where* $\omega = gcd(a, b)$, $gcd(r', s') = 1$ *and* χ *is the product of the prime factors of the form* $6k - 1$ *of* $gcd(d, q)$

(iii) χ *divides* c

(iv) $|\overrightarrow{\zeta}| = d\sqrt{2}$, $|\overrightarrow{\varsigma}| = d\sqrt{6}$, *and* $\overrightarrow{\zeta} \cdot \overrightarrow{\varsigma} = 0$.

(v) $s + i\sqrt{3}r = gcd(A - i\sqrt{3}B, 2q)$, *in the ring* $\mathbb{Z}[i\sqrt{3}]$, *where* $A = ac$ *and* $B = bd$.

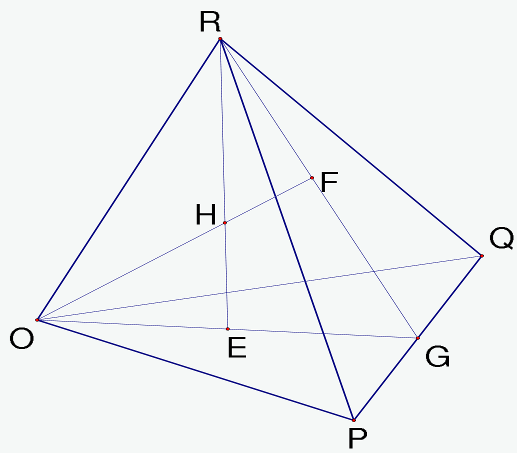
Example

If $d = 15$, we observe that we can take $a = 1$, $b = 7$ and $c = 25$ $(3d^2 = a^2 + b^2 + c^2)$. Then, $r = -5$, $s = 5$ give $\overrightarrow{\zeta} = (13, 16, -5)$ and $\overrightarrow{\eta} = (21, -3, 0)$.

$$\overrightarrow{OP} = m\overrightarrow{\zeta} - n\overrightarrow{\eta} = (13m - 21n, 16m + 3n, -5m)$$

$$\overrightarrow{OQ} = n\overrightarrow{\zeta} + (m - n)\overrightarrow{\eta} = (21m - 8n, -3m + 19n, -5n)$$

Regular Tetrahedra



$$\Omega(k) := \{(m,n) \in \mathbb{Z}^2 : m^2 - mn + n^2 = k^2\}$$

$$R = \left(\frac{\begin{array}{c}(2\zeta_1 - \eta_1)m \\ -(\zeta_1 + \eta_1)n \\ \pm 2ak\end{array}}{3}, \frac{\begin{array}{c}(2\zeta_2 - \eta_2)m \\ -(\zeta_2 + \eta_2)n \\ \pm 2bk\end{array}}{3}, \frac{\begin{array}{c}(2\zeta_3 - \eta_3)m \\ -(\zeta_3 + \eta_3)n \\ \pm 2ck\end{array}}{3} \right), \ (m,n) \in \Omega(k). \tag{8}$$

Theorem C (JNT 2009)

Every regular tetrahedron in $\mathbb{Z}^3$ *having one of its vertices the origin and side lengths* $\lambda\sqrt{2}$, *can be obtained by taking as one of its faces an equilateral triangle described by the previous parametrization in which with* a, b, c *and* d *odd integers satisfying* $a^2+b^2+c^2=3d^2$ *with* d *a divisor of* λ, *and then completing it with the fourth vertex as in (4) for some* $(m,n)\in\Omega(\frac{\lambda}{d})$.

Conversely, if we let a, b, c *and* d *be a primitive solution of* $a^2+b^2+c^2=3d^2$, *let* $k\in\mathbb{N}$ *and* $(m,n)\in\Omega(k)$, *then the coordinates of the point* R *in (4) are*

(i) all integers, if $k\equiv 0 \pmod 3$ *regardless of the choice of signs or*

(ii) integers, precisely for only one choice of the signs if $k\not\equiv 0 \pmod 3$.

Examples of $k^2 = m^2 - mn + n^2$, $gcd(m, n) = 1$, $2m < n$

[7, [3, 8]]   [13, [7, 15]]   [19, [5, 21]]   [31, [11, 35]]   [37, [7, 40]]

[43, [13, 48]]   [49, [16, 55]]   [61, [9, 65]]   [67, [32, 77]]

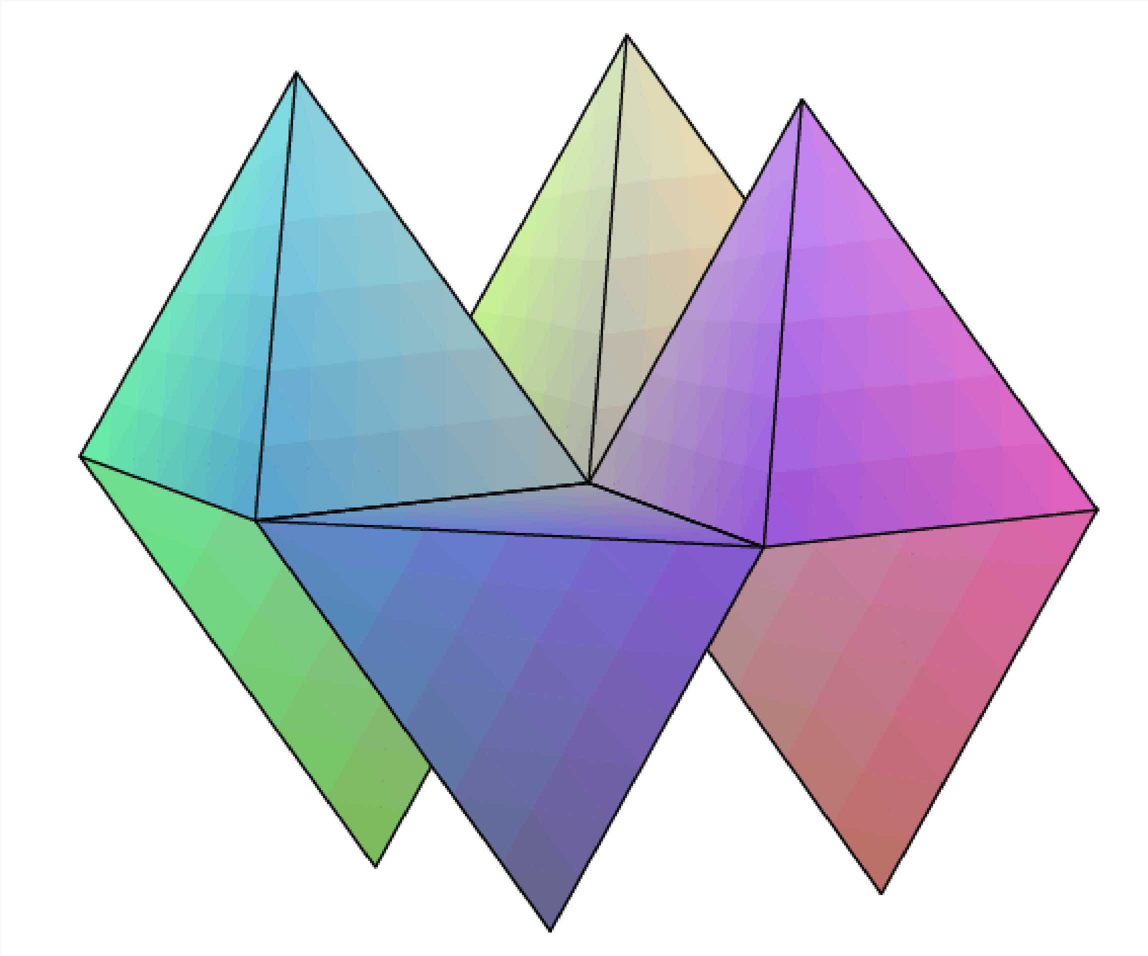
[73, [17, 80]]   [79, [40, 91]] $[91, [11, 96], [19, 99]]$ [97, [55, 112]] *The number of regular tetrahedra whose coordinates of its vertices are in the set* $\{0, 1, ..., n\}$ *is the sequence* $2A103158$ **Link:** A103158 (2005)

n	1	2	3	4	5	6	7	8	9	10	11
A103158	1	9	36	104	257	549	1058	1896	3199	5154	7926

n	12	13	14	15	16	17	18
A103158	11768	16967	23859	32846	44378	58977	77215

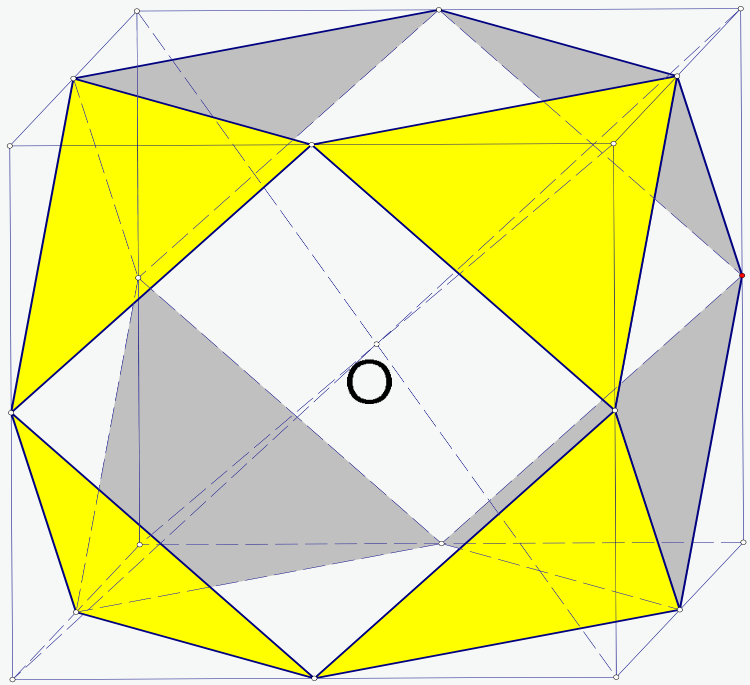
n	19	20	21	22	23	24	25
A103158	99684	126994	159963	199443	246304	301702	366729

n	26	27	28	29	30	31	32
A103158	442587	530508	631820	1748121	880941	31031930	1202984



Why this pattern?

Cuboctahedron, $d = 1$



$\boxed{1}$ 1,1, $\{[0,0,0],[1,1,0],[1,0,1],[0,1,1]\}$

$N := \{[[-1,-1,1],1],[[1,-1,-1],1],[[1,-1,1],1],[[-1,-1,-1],1]\}$

$\boxed{2}$ 3, 4, $\{[1,1,0],[0,0,4],[4,1,3],[1,4,3]\}$

$N := \{[[-1,-1,1],1],[[-1,5,1],3],[[-5,1,-1],3],[[1,1,5],3]\}$

$\boxed{3}$ 5, 7, $\{[0,0,4],[7,0,3],[3,5,0],[4,5,7]\}$,

$N := \{[[1,5,7],5],[[7,-5,-1],5],[[1,-5,7],5],[[7,5,-1],5]\}$

$\boxed{4}$ 7, 9, $\{[9,0,9],[0,4,8],[8,9,5],[5,1,0]\}$

$N := \{[[-5,-11,1],7],[[-1,1,-1],1],[[-1,-5,-11],7],[[-11,1,5],7]\}$

$\boxed{5}$ 9, 12, $\{[11,9,0],[11,0,9],[0,5,5],[8,12,12]\}$

$N := \{[[-1,-11,-11],9],[[-7,13,-5],9],[[-5,-1,-1],3],[[7,5,-13],9]\}$

$\boxed{6}$ 11, 15, $\{[4,0,0],[7,13,8],[15,0,11],[0,1,15]\}$

$N := \{[[13,5,-13],11],[[-1,-19,1],11],[[17,-7,5],11],[[-5,-7,-17],11]\}$

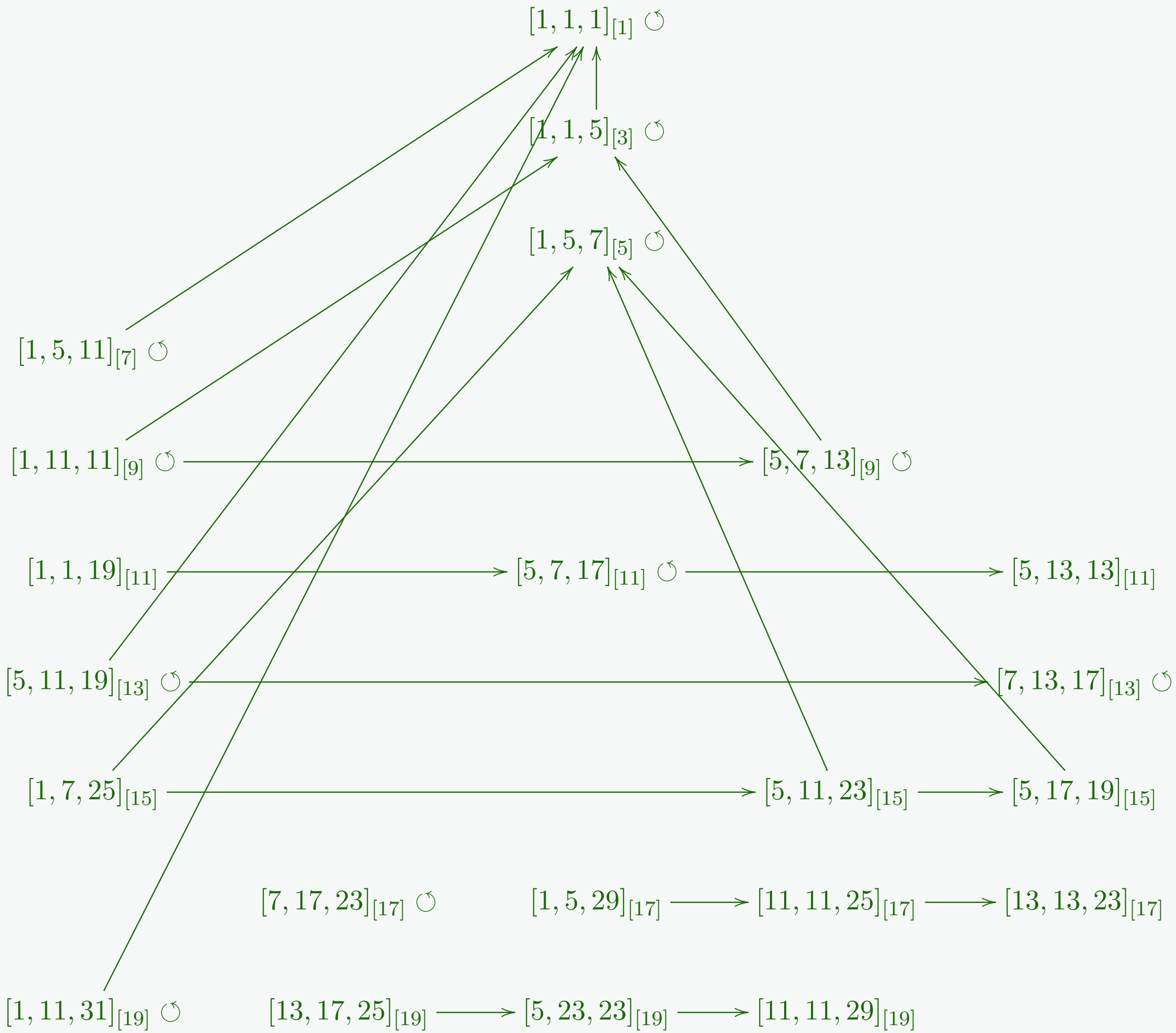
$\boxed{7}$ 13, 16, $\{[0,9,15],[15,16,7],[16,0,16],[7,1,0],[5,11,19]\}$

$N := \{[[-5,-11,-19],13],[[11,19,-5],13],[[19,-5,-11],13],[[-1,1,-1],1]\}$

$\boxed{8}$ 13, 17, $\{[17,13,5],[0,13,12],[5,0,0],[12,0,17],[7,13,17]]\}$

$N := \{[[-17,-13,7],13],[[7,-13,17],13],[[-17,13,7],13],[[7,13,17],13]\}$

Graph RT



[1, 1, 1][1] ↻
[1, 1, 5][3] ↻
[1, 5, 7][5] ↻
[1, 5, 11][7] ↻
[1, 11, 11][9] ↻
[5, 7, 13][9] ↻
[1, 1, 19][11]
[5, 7, 17][11] ↻
[5, 13, 13][11]
[5, 11, 19][13] ↻
[7, 13, 17][13] ↻
[1, 7, 25][15]
[5, 11, 23][15]
[5, 17, 19][15]
[7, 17, 23][17] ↻
[1, 5, 29][17]
[11, 11, 25][17]
[13, 13, 23][17]
[1, 11, 31][19] ↻
[13, 17, 25][19]
[5, 23, 23][19]
[11, 11, 29][19]

Example with all four normals with differnet d's

The regular tetrahedron $OABC$ where $O = (0, 0, 0)$,

$$A = (376, -841, 2265), B = (-1005, -2116, 701), C = (1411, -1965, 356)$$

has the four faces with normal vectors.

$$(-187, 113, 73), \quad \text{satisfying} \quad 187^2 + 113^2 + 73^2 = 3(133^2),$$
$$(-343, -253, -37), \quad \text{satisfying} \quad 343^2 + 253^2 + 37^2 = 3(247)^2,$$
$$(19, 41, 151), \quad \text{satisfying} \quad 19^2 + 41^2 + 151^2 = 3(91)^2 \quad \text{and}$$
$$(391, -2461, 1661), \quad \text{satisfying} \quad 391^2 + 2461^2 + 1661^2 = 3(1729)^2.$$

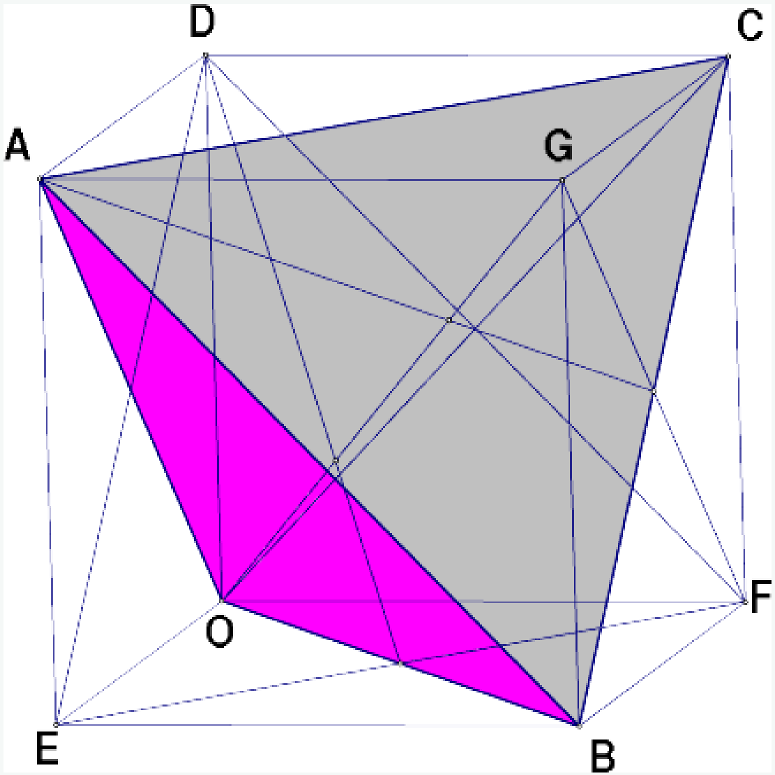
Example 2: The points $O = (0, 0, 0)$, $A = [-6677, -2672, 1445]$,

$B = [-5940, 4143, -1167]$, $C = [-3837, 2595, 5688]$ form a regular tetrahedron of side-lengths equal to $5187\sqrt{2}$ and the highest d for its faces is 1729.

Observation: $5187 = (3)(7)(13)(19)$ and the primes 3, 7, 13 and 19 are first primes of the form $x^2 + 3y^2$, $x, y \in \mathbb{Z}$.

The cubes in $\mathbb{Z}^3$

Theorem D *Every cube in $\mathbb{Z}^3$ can be obtained by a translation along a vector with integer coordinates from a cube with a vertex the origin containing a regular tetrahedron with a vertex at the origin and all integer coordinates (see figure below) and as a result it must have side lengths equal to n for some $n \in \mathbb{N}$. Conversely, given a regular tetrahedron in $\mathbb{Z}^3$, this can be completed to a cube which is going to be automatically in $\mathbb{Z}^3$.*



The number of cubes whose coordinates for its vertices are in the set $\{0,1,...,n\}$ *is the sequence A098928:*

n	1	2	3	4	5	6	7	8	9	10	11
A098928	1	9	36	100	229	473	910	1648	2795	4469	6818

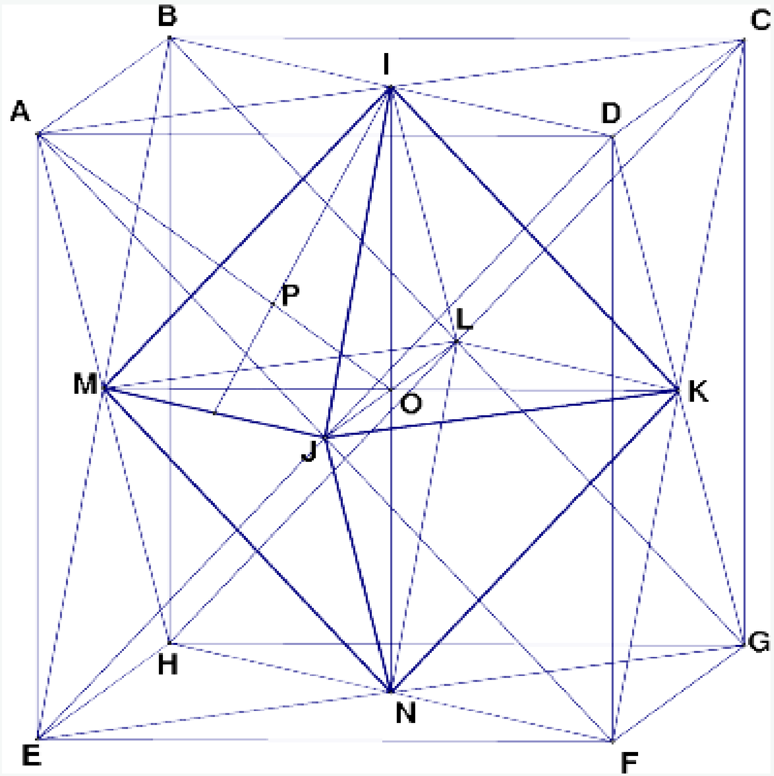
n	12	13	14	15	16	17	18
A098928	10032	14315	19907	27190	36502	48233	62803

.

$$A098928 \leq A103158$$

The octahedrons in $\mathbb{Z}^3$

Theorem E *Every regular octahedrons in $\mathbb{Z}^3$ is the dual of a cube that can be obtained (up to a translation with a vector with integer coordinates) by doubling a cube in $\mathbb{Z}^3$.*



The number of regular octahedrons whose coordinates for its vertices are in the set $\{0, 1, ..., n\}$ *denoted by* $\mathcal{RO}(n)$ *:*

n	1	2	3	4	5	6	7	8	9	10	11
$\mathcal{RO}(n)$	0	1	8	32	104	261	544	1000	1696	2759	4296

n	12	13	14	15	16	17	18
$\mathcal{RO}(n)$	6434	9352	13243	18304	24774	32960	43223

.

Orthogonal matrices with rational coefficients

$$T_3 := \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ -2 & -2 & -1 \end{bmatrix}, T_5 := \frac{1}{5}\begin{bmatrix} 4 & 0 & 3 \\ 3 & 0 & -4 \\ 0 & -5 & 0 \end{bmatrix}, T_7 := \frac{1}{7}\begin{bmatrix} -2 & -3 & 6 \\ 3 & -6 & -2 \\ -6 & -2 & -3 \end{bmatrix}, T_9 := \frac{1}{9}\begin{bmatrix} -7 & -4 & 4 \\ 4 & 1 & 8 \\ -4 & 8 & 1 \end{bmatrix}$$

$$T_{11} := \frac{1}{11}\begin{bmatrix} 2 & -9 & -6 \\ 9 & -2 & 6 \\ -6 & -6 & 7 \end{bmatrix}, T_{13} := \frac{1}{13}\begin{bmatrix} -4 & -12 & -3 \\ 12 & -3 & -4 \\ 3 & -4 & 12 \end{bmatrix}, \hat{T}_{13} := \frac{1}{13}\begin{bmatrix} 0 & -13 & 0 \\ 12 & 0 & 5 \\ -5 & 0 & 12 \end{bmatrix}.$$

$$T_{17} := \frac{1}{17}\begin{bmatrix} 12 & -8 & -9 \\ 12 & 9 & 8 \\ 1 & -12 & 12 \end{bmatrix}, \hat{T_{17}} := \frac{1}{17}\begin{bmatrix} 15 & 0 & 8 \\ 8 & 0 & -15 \\ 0 & -17 & 0 \end{bmatrix},$$

$$T_{19} := \frac{1}{19}\begin{bmatrix} 6 & -18 & 1 \\ 17 & 6 & 6 \\ -6 & -1 & 18 \end{bmatrix}, \hat{T}_{19} := \frac{1}{19}\begin{bmatrix} 15 & -6 & -10 \\ 10 & 15 & 6 \\ 6 & -10 & 15 \end{bmatrix}.$$

Good source of finite subgroups $GL(3, \mathbb{Z}_p)$ *by taking convenient primes.*

Conjectures and Problems

[1] *The Diophantine equation* $a^2 + b^2 + c^2 = 3d^2$ *has degenerate solutions, i.e.* $gcd(a, b, c) = 1$, $gcd(a, d) > 1$, $gcd(b, d) > 1$ *and* $gcd(a, d) > 1$, *if and only if* d *has at least three distinct prime factors of the form* $4k+1$, $k \in \mathbb{N}$.

[2] *The graph RT has infinitely many connected components and a fractal structure.*

[3] *The tetrahedron* $OABC$, *with* $O = (0, 0, 0)$, $A = [-6677, -2672, 1445]$, $B = [-5940, 4143, -1167]$, $C = [-3837, 2595, 5688]$, *gives the smallest side between all sides* $\ell\sqrt{2}$ *of irreducible regular tetrahedra with the property that their faces have equations* $a_i^2 + b_i^2 + c_i^2 = 3d_i^2$ *with* $d_i < \ell$.

[4] *What is the number of ireducible cubes in* $\mathbb{Z}^3$ *with sides lengths* d *(odd) modulo the cube symmetries?*

References

[1] *N. C. Ankeny,* Sums of Three Squares, *Proceedings of AMS, vol. 8, No. 2, pp 316-319.*

[2] *R. Chandler and E. J. Ionascu, A characterization of all equilateral triangles in* $\mathbb{Z}^3$, *Integers, Art. A19 of Vol.* **8** *(2008)*

[3] *S. Cooper and M. Hirschhorn, On the number of primitive representations of integers as a sum of squares,* Ramanujan J. *(2007) 13, pp. 7-25.*

[4] *D. A. Cox, Primes of the Form* $x^2 + ny^2$*: Fermat, Class Field Theory, and Complex Multiplication, Wiley-Interscience, 1997*

[5] *R. Guy,* Unsolved Problems in Number Theory, *Springer-Verlag, 2004*

[6] *E. Grosswald, Representations of integers as sums of squares, Springer Verlag, New York, 1985.*

[7] *I. J. Schoenberg, Regular Simplices and Quadratic Forms, J. London Math. Soc. 12 (1937) 48-55.*

[8] *M. D. Hirschhorn and J. A. Sellers, On representations of numbers as a sume of three squares,* Discrete Mathematics, *199 (1999), pp. 85-101.*

[9] *E. J. Ionascu,* A parametrization of equilateral triangles having integer coordinates, *Journal of Integer Sequences, Vol.* **10**, *09.6.7. (2007)*

[10] *E. J. Ionascu,* A characterization of regular tetrahedra in $\mathbb{Z}^3$, *J. Number Theory, 129(2009), 1066-1074.*

[11] *E. J. Ionascu and A. Markov, Platonic solids in* $\mathbb{Z}^3$, *J. Number Theory, 131 (2011), pp. 138-145.*

[12] *E. J. Ionascu and R. Obando, Cubes in* $\{0, 1, ..., n\}^3$, *to appear in Integers 2012.*

[13] *K. Rosen,* Elementary Number Theory, *Fifth Edition, Addison Wesley, 2004*

[14] *Neil J. A. Sloane,* The On-Line Encyclopedia of Integer Sequences, *2005, published electronically at http://www.research.att.com/∼njas/sequences/.*