Existence of regular polyhedrons in space with integer coordinates

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Two examples

**Figure 1a,** \( \{(0,0,4),(7,0,3),(3,5,0),(4,5,7)\} \)

**Figure 1b,** \( \{(1,4,3),(3,3,1),(1,1,0),(−1,2,2),(2,2,5),(4,1,3),(2,−1,2),(0,0,4)\} \)

Euclidean Distance: \( d(A,B) := \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2 + (a_z - b_z)^2} \), where \( A = (a_x,a_y,a_z) \) and \( B = (b_x,b_y,b_z) \)
An example of a regular octahedron in $\mathbb{Z}^3$:

*Figure 2, $\{(0,0,0),(1,-16,9),(-8,-15,-7),(16,-9,1),(7,-8,-15),(8,-24,-6)\}$*
**Theorem A** (JNT 2011, A. Markov) *There is no regular icosahedron or regular dodecahedron in \( \mathbb{Z}^3 \).*
The lattice \( \mathcal{P}_{a,b,c} \) containing equilateral triangles

\[
\mathcal{P}_{a,b,c} := \{(\alpha, \beta, \gamma) \in \mathbb{Z}^3| \quad a\alpha + b\beta + c\gamma = e, \quad a^2 + b^2 + c^2 = 3d^2, \quad a, b, c, d, e \in \mathbb{Z}\}.
\]
\[ u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = a \vec{i} + b \vec{j} + c \vec{k}, \quad a, b, c \in \mathbb{Z} \]
\[ |u \times v| = 2 \frac{|u| \cdot |v| \sin(60^\circ)}{2} = \sqrt{3} \frac{\ell^2}{2} \implies \]

\[ |u \times v|^2 = a^2 + b^2 + c^2 = 3 \left( \frac{\ell^2}{2} \right)^2 \implies \]

\( \ell^2 \) must be divisible by 2 and if \( \ell^2 / 2 = d \) we have

\[ a^2 + b^2 + c^2 = 3d^2. \]
Fact 1: The cosine of the dihedral angle between two adjacent faces of a regular icosahedron is \(-\frac{\sqrt{5}}{3}\). (Exercise)

Fact 2: The cosine of the dihedral angle between two planes \(P_{a,b,c}\) and \(P_{a',b',c'}\) is

\[
\cos \alpha = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|} = \frac{aa' + bb' + cc'}{3dd'}
\]

Observation: For a regular tetrahedron the cosine of the dihedral angle is \(\frac{1}{3}\), for the cube is 0, and for regular octahedron is \(-\frac{1}{3}\).
Examples of planes $P_{a,b,c}$

\[
\begin{align*}
d & = 1, \ [1, 1, 1] \\
& = 3, \ [1, 1, 5] \\
& = 5, \ [1, 5, 7] \\
& = 7, \ [1, 5, 11] \\
& = 9, \ [1, 11, 11] [5, 7, 13] \\
& = 11, \ [1, 1, 19], [5, 7, 17], [5, 13, 13] \\
& = 13, \ [5, 11, 19], [7, 13, 17] \\
& = 15, \ [1, 7, 25], [5, 11, 23], [5, 17, 19] \\
& = 17, \ [1, 5, 29], [7, 17, 23], [11, 11, 25], [13, 13, 23] \\
& = 19, \ [1, 11, 31], [5, 23, 23], [11, 11, 29], [13, 17, 25] \\
& = 21, \ [1, 19, 31], [11, 19, 29], [13, 23, 25]
\end{align*}
\]

**Question:** What is the number of primitive solutions $(a,b,c)$, given an odd positive integer $d$? We will denote this number by $\pi\epsilon(d)$. 
Graph of $k \rightarrow \pi \epsilon (2k + 1), k = 1, 2, ..., 10000$

Slope of the average line $m \approx 0.34131$ computed for the first 10 million odd numbers.
\[ \pi \epsilon (d) = \frac{\Lambda(d) + 24 \Gamma_2(3d^2)}{48} \]  \hspace{1cm} (2)

\[ \Lambda(d) := 8d \prod_{\substack{p | d, p \text{ prime}}} \left( 1 - \frac{(-3)}{p} \right) \]  \hspace{1cm} (3)

Legendre symbol

\[ \left( \frac{-3}{p} \right) = \begin{cases} 
0 & \text{if } p = 3 \\
1 & \text{if } p \equiv 1 \text{ or } 7 \pmod{12} \\
-1 & \text{if } p \equiv 5 \text{ or } 11 \pmod{12}
\end{cases} \]  \hspace{1cm} (4)

\[ \Gamma_2(d) = \begin{cases} 
0 & \text{if } d \text{ is divisible by a prime factor of the form } 8s+5 \text{ or } 8s+7, \ s \geq 0, \\
2^{k-1} & \text{where } k \text{ is the number of distinct prime factors of } d \\
\text{of } d \text{ of the form } 8s+1, \text{ or } 8s+3 \ (s \geq 0)
\end{cases} \]  \hspace{1cm} (5)
Theorem B: The sub-lattice $\mathcal{P}_{eq,a,b,c}$ is generated by two vectors $\vec{\zeta}$ and $\vec{\eta}$ in the following sense: $\mathcal{T}_{a,b,c}^{m,n} := \triangle OPQ$ with $P, Q$ in $\mathcal{P}_{a,b,c}$, is equilateral if and only if for some integers $m, n$

$$\overrightarrow{OP} = m \vec{\zeta} - n \vec{\eta}, \quad \overrightarrow{OQ} = n \vec{\zeta} + (m - n) \vec{\eta},$$

with $\vec{\zeta} = (\zeta_1, \zeta_1, \zeta_2)$, $\vec{\varsigma} = (\varsigma_1, \varsigma_2, \varsigma_3)$, $\vec{\eta} = \frac{\vec{\zeta} + \vec{\varsigma}}{2}$, \quad (6)

$$\begin{cases}
\zeta_1 = -\frac{rac + dbs}{q} \\
\zeta_2 = \frac{das - bcr}{q} \\
\zeta_3 = r \\
\varsigma_1 = \frac{3dbr - acs}{q} \\
\varsigma_2 = -\frac{3dar + bcs}{q} \\
\varsigma_3 = s
\end{cases} \quad , \quad \begin{cases}
\varsigma_1 = \frac{3dbr - acs}{q} \\
\varsigma_2 = -\frac{3dar + bcs}{q} \\
\varsigma_3 = s
\end{cases} \quad , \quad (7)
$$

where $q = a^2 + b^2$ and $r, s$ can be chosen so that all six numbers in (3) are odd integers. The sides-lengths of $\triangle OPQ$ are equal to $d\sqrt{2(m^2 - mn + n^2)}$.

Moreover, $r$ and $s$ can be constructed in such a way that the following properties are also verified:

(i) $r$ and $s$ satisfy $2q = s^2 + 3r^2$ and as a result $2(b^2 + c^2) = \varsigma_1^2 + 3\varsigma_2^2$ and $2(a^2 + c^2) = \varsigma_2^2 + 3\varsigma_2^2$

(ii) $r = r'\omega \chi$, $s = s'\omega \chi$ where $\omega = \gcd(a, b)$, $\gcd(r', s') = 1$ and $\chi$ is the product of the prime factors of the form $6k - 1$ of $\gcd(d, q)$

(iii) $\chi$ divides $c$

(iv) $|\vec{\zeta}| = d\sqrt{2}$, $|\vec{\varsigma}| = d\sqrt{6}$, and $\vec{\zeta} \cdot \vec{\varsigma} = 0$.

(v) $s + i\sqrt{3}r = \gcd(A - i\sqrt{3}B, 2q)$, in the ring $\mathbb{Z}[i\sqrt{3}]$, where $A = ac$ and $B = bd$. 
Example

If \( d = 15 \), we observe that we can take \( a = 1 \), \( b = 7 \) and \( c = 25 \) \((3d^2 = a^2 + b^2 + c^2)\). Then, \( r = -5 \), \( s = 5 \) give \( \vec{\zeta} = (13, 16, -5) \) and \( \vec{\eta} = (21, -3, 0) \).

\[
\overrightarrow{OP} = m \vec{\zeta} - n \vec{\eta} = (13m - 21n, 16m + 3n, -5m)
\]

\[
\overrightarrow{OQ} = n \vec{\zeta} + (m - n) \vec{\eta} = (21m - 8n, -3m + 19n, -5n)
\]
Regular Tetrahedra

\[ \Omega(k) := \{(m, n) \in \mathbb{Z}^2 : m^2 - mn + n^2 = k^2\} \]

\[ R = \begin{pmatrix} (2\zeta_1 - \eta_1)m & (2\zeta_2 - \eta_2)m & (2\zeta_3 - \eta_3)m \\ -(\zeta_1 + \eta_1)n & -(\zeta_2 + \eta_2)n & -(\zeta_3 + \eta_3)n \\ \pm 2ak & \pm 2bk & \pm 2ck \end{pmatrix} \frac{3}{3}, \frac{3}{3}, \frac{3}{3}, (m, n) \in \Omega(k). \]
Theorem C (JNT 2009)

Every regular tetrahedron in $\mathbb{Z}^3$ having one of its vertices the origin and side lengths $\lambda \sqrt{2}$, can be obtained by taking as one of its faces an equilateral triangle described by the previous parametrization in which with $a$, $b$, $c$ and $d$ odd integers satisfying $a^2 + b^2 + c^2 = 3d^2$ with $d$ a divisor of $\lambda$, and then completing it with the fourth vertex as in (4) for some $(m, n) \in \Omega(\frac{\lambda}{d})$.

Conversely, if we let $a$, $b$, $c$ and $d$ be a primitive solution of $a^2 + b^2 + c^2 = 3d^2$, let $k \in \mathbb{N}$ and $(m, n) \in \Omega(k)$, then the coordinates of the point $R$ in (4) are

(i) all integers, if $k \equiv 0 \pmod{3}$ regardless of the choice of signs or

(ii) integers, precisely for only one choice of the signs if $k \not\equiv 0 \pmod{3}$. 
Examples of $k^2 = m^2 - mn + n^2$, $gcd(m, n) = 1$, $2m < n$

[7, [3, 8]]  [13, [7, 15]]  [19, [5, 21]]  [31, [11, 35]]  [37, [7, 40]]
[43, [13, 48]]  [49, [16, 55]]  [61, [9, 65]]  [67, [32, 77]]
[73, [17, 80]]  [79, [40, 91]]  [91, [11, 96], [19, 99]]  [97, [55, 112]]
The number of regular tetrahedra whose coordinates of its vertices are in the set \( \{0, 1, \ldots, n\} \) is the sequence 2A103158

**Link:** A103158 (2005)

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Why this pattern?
Cuboctahedron, $d = 1$
\[ 1, 1, \{[0, 0, 0], [1, 1, 0], [1, 0, 1], [0, 1, 1]\} \]

\[ N := \{[-1, -1, 1], [1, -1, -1], [1, -1, 1], [-1, -1, -1], 1\} \]

\[ 2, 3, 4, \{[1, 1, 0], [0, 0, 4], [4, 1, 3], [1, 4, 3]\} \]

\[ N := \{[-1, -1, 1], [-1, 5, 1], [-5, 1, -1], [1, 1, 5], 3\} \]

\[ 3, 5, 7, \{[0, 0, 4], [7, 0, 3], [3, 5, 0], [4, 5, 7]\} \]

\[ N := \{[1, 5, 7], 5, [7, -5, -1], 5, [1, -5, 7], 5, [7, 5, -1], 5\} \]

\[ 4, 7, 9, \{[9, 0, 9], [0, 4, 8], [8, 9, 5], [5, 1, 0]\} \]

\[ N := \{[-5, -11, 1], 7, [-1, 1, -1], 1, [-1, -5, -11], 7, [-11, 1, 5], 7\} \]

\[ 5, 9, 12, \{[11, 9, 0], [11, 0, 9], [0, 5, 5], [8, 12, 12]\} \]

\[ N := \{[-1, -11, -11], 9, [-7, 13, -5], 9, [-5, -1, -1], 3, [7, 5, -13], 9\} \]

\[ 6, 11, 15, \{[4, 0, 0], [7, 13, 8], [15, 0, 11], [0, 1, 15]\} \]

\[ N := \{[13, 5, -13], 11, [-1, -19, 1], 11, [17, -7, 5], 11, [-5, -7, -17], 11\} \]

\[ 7, 13, 16, \{[0, 9, 15], [15, 16, 7], [16, 0, 16], [7, 1, 0], [5, 11, 19]\} \]

\[ N := \{[-5, -11, -19], 13, [11, 19, -5], 13, [19, -5, -11], 13, [-1, 1, -1], 1\} \]

\[ 8, 13, 17, \{[17, 13, 5], [0, 13, 12], [5, 0, 0], [12, 0, 17], [7, 13, 17]\} \]

\[ N := \{[-17, -13, 7], 13, [7, -13, 17], 13, [-17, 13, 7], 13, [7, 13, 17], 13\} \]
Graph $RT$
Example with all four normals with different d’s

The regular tetrahedron $OABC$ where $O = (0, 0, 0),$ 

$$A = (376, -841, 2265), B = (-1005, -2116, 701), C = (1411, -1965, 356)$$

has the four faces with normal vectors.

$$(-187, 113, 73), \text{ satisfying } 187^2 + 113^2 + 73^2 = 3(133^2),$$

$$(-343, -253, -37), \text{ satisfying } 343^2 + 253^2 + 37^2 = 3(247)^2,$$

$$(19, 41, 151), \text{ satisfying } 19^2 + 41^2 + 151^2 = 3(91)^2 \text{ and}$$

$$(391, -2461, 1661), \text{ satisfying } 391^2 + 2461^2 + 1661^2 = 3(1729)^2.$$
Theorem D  Every cube in $\mathbb{Z}^3$ can be obtained by a translation along a vector with integer coordinates from a cube with a vertex the origin containing a regular tetrahedron with a vertex at the origin and all integer coordinates (see figure below) and as a result it must have side lengths equal to $n$ for some $n \in \mathbb{N}$. Conversely, given a regular tetrahedron in $\mathbb{Z}^3$, this can be completed to a cube which is going to be automatically in $\mathbb{Z}^3$. 
The number of cubes whose coordinates for its vertices are in the set \( \{0, 1, \ldots, n\} \) is the sequence A098928:

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\( A098928 \leq A103158 \)

The octahedrons in \( \mathbb{Z}^3 \)
Theorem E  Every regular octahedrons in $\mathbb{Z}^3$ is the dual of a cube that can be obtained (up to a translation with a vector with integer coordinates) by doubling a cube in $\mathbb{Z}^3$. 
The number of regular octahedrons whose coordinates for its vertices are in the set \( \{0, 1, \ldots, n\} \) denoted by \( \text{RO}(n) \):

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Orthogonal matrices with rational coefficients

\[ T_3 := \frac{1}{3} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ -2 & -2 & -1 \end{bmatrix} , \ T_5 := \frac{1}{5} \begin{bmatrix} 4 & 0 & 3 \\ 3 & 0 & -4 \\ 0 & -5 & 0 \end{bmatrix} , \ T_7 := \frac{1}{7} \begin{bmatrix} -2 & -3 & 6 \\ 3 & -6 & -2 \\ -6 & -2 & -3 \end{bmatrix} , \ T_9 := \frac{1}{9} \begin{bmatrix} -7 & -4 & 4 \\ 4 & 1 & 8 \\ -4 & 8 & 1 \end{bmatrix} \]

\[ T_{11} := \frac{1}{11} \begin{bmatrix} 2 & -9 & -6 \\ 9 & -2 & 6 \\ -6 & -6 & 7 \end{bmatrix} , \ T_{13} := \frac{1}{13} \begin{bmatrix} -4 & -12 & -3 \\ 12 & -3 & -4 \\ 3 & -4 & 12 \end{bmatrix} , \ \hat{T}_{13} := \frac{1}{13} \begin{bmatrix} 0 & -13 & 0 \\ 12 & 0 & 5 \\ -5 & 0 & 12 \end{bmatrix} . \]

\[ T_{17} := \frac{1}{17} \begin{bmatrix} 12 & -8 & -9 \\ 12 & 9 & 8 \\ 1 & -12 & 12 \end{bmatrix} , \ \hat{T}_{17} := \frac{1}{17} \begin{bmatrix} 15 & 0 & 8 \\ 8 & 0 & -15 \\ 0 & -17 & 0 \end{bmatrix} , \ T_{19} := \frac{1}{19} \begin{bmatrix} 6 & -18 & 1 \\ 17 & 6 & 6 \\ -6 & -1 & 18 \end{bmatrix} , \ \hat{T}_{19} := \frac{1}{19} \begin{bmatrix} 15 & -6 & -10 \\ 10 & 15 & 6 \\ 6 & -10 & 15 \end{bmatrix} . \]

Good source of finite subgroups $GL(3, \mathbb{Z}_p)$ by taking convenient primes.
Conjectures and Problems

1. The Diophantine equation $a^2 + b^2 + c^2 = 3d^2$ has degenerate solutions, i.e. $\gcd(a, b, c) = 1$, $\gcd(a, d) > 1$, $\gcd(b, d) > 1$ and $\gcd(a, d) > 1$, if and only if $d$ has at least three distinct prime factors of the form $4k+1$, $k \in \mathbb{N}$.

2. The graph $RT$ has infinitely many connected components and a fractal structure.

3. The tetrahedron $OABC$, with $O = (0, 0, 0)$, $A = [-6677, -2672, 1445]$, $B = [-5940, 4143, -1167]$, $C = [-3837, 2595, 5688]$, gives the smallest side between all sides $\ell\sqrt{2}$ of irreducible regular tetrahedra with the property that their faces have equations $a_i^2 + b_i^2 + c_i^2 = 3d_i^2$ with $d_i < \ell$.

4. What is the number of irreducible cubes in $\mathbb{Z}^3$ with sides lengths $d$ (odd) modulo the cube symmetries?
References


