Ehrhart polynomial for lattice squares, cubes, and hypercubes

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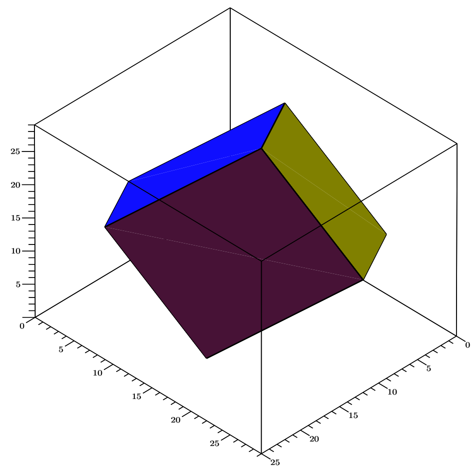
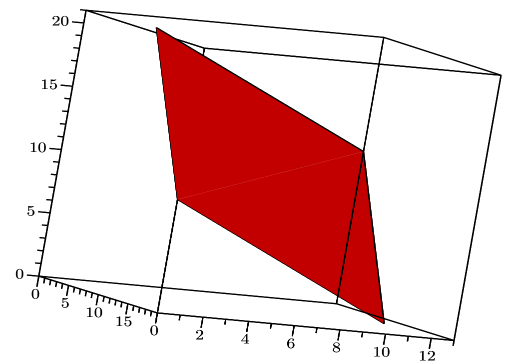
Abstract

We are investigating the problem of constructing integer lattice squares, cubes and hypercubes in Euclidean spaces of dimension less then or equal to four. We then study the number of lattice points inside of regions with such boundaries. For tools, we are using some results from the theory of Ehrhart polynomials and elementary number theory. Some preliminary results are presented and fresh conjectures will be shared.

The Ehrhart's polynomial, $L(P,t)$ counts how many lattice points are inside tP

$$L(P,t) = c_0 t^d + c_1 t^{d-1} + \ldots + c_{d-1} t + c_d$$





- d, c_0, c_1 and c_d

- $(-1)^d L(P,-t)$ (combinatorial reciprocity theorem)

• $L(P \times Q, t) = L(P,t)L(Q,t)$ (cross product of polytopes)

• **Exercise:** The "volume" of the fundamental domain of the sub-lattice $\mathbb{Z}^d$ of all solutions $(x_1, x_2, ...x_d) \in \mathbb{Z}^d$ satisfying

$$a_1x_1 + a_2x_2 + ... + a_dx_d = 0$$

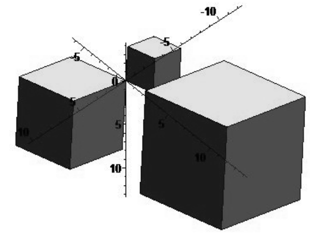
is equal to $\sqrt{a_1^2 + a_2^2 + ... + a_d^2}$, where $a_1, a_2, a_3, ..., a_d \in \mathbb{Z}$ and $\gcd(a_1, a_2, ..., a_d) = 1$.

• M. Beck and S. Robins, *Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, (2015)

• A. Barvinok, *Integer Points in Polyhedra*, European Mathematical Society, (2008)

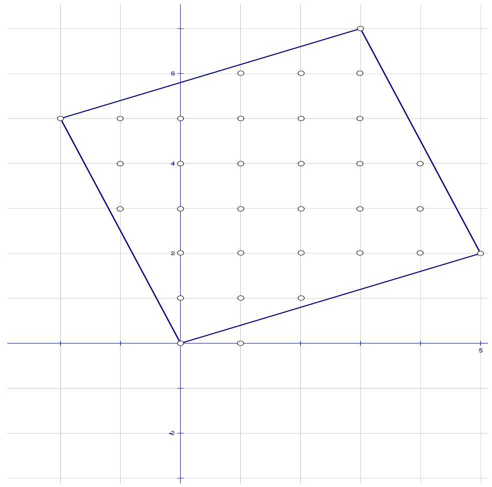
• S. Sam, *A bijective proof for a theorem of Ehrhart*, Amer. Math. Monthly 116 (2009), no. 8, pp. 688-701

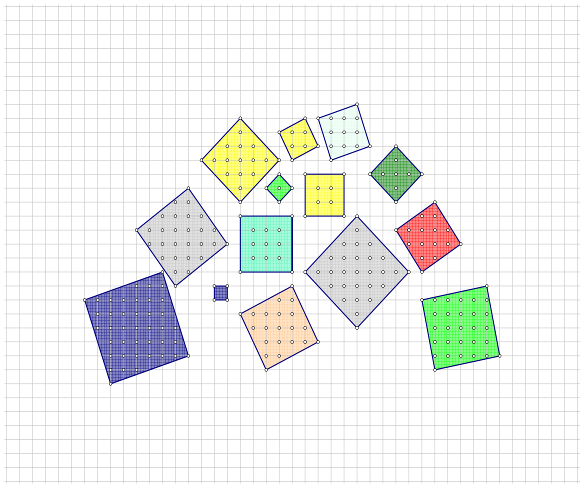
• R. Diaz and S. Robins, *The Ehrhart polynomial of a lattice polytope*, Ann. of Math. 145 (1997) pp. 503-518



• **Eugène** Ehrhart, *Sur les polyèdres rationnels homothétiques à n dimensions*, C. R. Acad. Sci. Paris 254 (1962), 616–618

Sequence of almost perfect squares in two dimensions





0, 1, 4, 5, 9, 12, 13, 16, 17, 24, 25, 28, 33, 36, 37, 40, 41, 49, 52, 57, 60, 61, 64, 65, 72, 73, 81, 84, 85, 88, 96, 97, 100,..., A194154

$\ldots \nearrow 2015, \searrow 2016$

Given a, b with $\gcd(a,b)=1$, then

$$E_{\square}(T) = (a^2+b^2)t^2 + 2t + 1, \ \ t \in \mathbb{N}.$$

Problem 1: *Find the number of integer solutions* (x,y) *of the system*

$$4x+3y, 4y-3x \in [1,99].$$

Problem 2: *Find the number of integer solutions* (x,y) *of the system*

$$4x+5y, 4y-5x \in [1,81].$$

Theorem: *Given* a, b *with* $\mathrm{gcd}(a,b) = 1$ *and* $t \in \mathbb{N}$, *then*

$$\sharp\{(x,y) \in \mathbb{Z}^2 : \; ax + by, ay - bx \in [1, t(a^2 + b^2) - 1]\} =$$

$$(a^2 + b^2)t^2 - 2t + 1.$$

Dimension 3 (squares)

$u = (a, b, c)$, $v = (a', b', c')$, $a, b, c, a', b', c' \in \mathbb{Z}$ such that $a^2 + b^2 + c^2 = a'^2 + b'^2 + c'^2 = \ell$ and $aa' + bb' + cc' = 0$.

$\diamondsuit$ $\gcd(a, b, c, a', b', c') = 1$

$\diamondsuit$ $d = \gcd(a, b, c)$, $d' = gcd(a', b', c')$ and

$\diamondsuit$ $D = \gcd(bc' - b'c, ac' - a'c, ab' - b'a)$

Theorem: *The Ehrhart polynomial of a lattice square embedded into* $\mathbb{R}^3$*, described above and with the notation introduced is given by*

$$E_{\square}(t) = Dt^2 + (d + d')t + 1.$$

Proof:

Examples

♡ $u = (3, -3, 0)$ and $v = (1, 1, 4)$

♡ $u = 5(8, 12, 9)$ and $v = 17(0, -3, 4)$

$$\boxed{\ell^2 = a^2 + b^2 + c^2,\ \ a, b, c, \ell \in \mathbb{Z}}$$

ℓ	[a,b,c], gcd$(a, b, c) = 1$, $0 < a \le b \le c$
1	[1,0,0]
3	[1,2,2]
5	[3,4,0]
7	[3,6,2]
9	[1, 4, 8], [7, 4, 4]

ℓ	[a,b,c], gcd$(a, b, c) = 1$
11	[9, 2, 6], [7, 6, 6]
13	[5, 12, 0], [3, 4, 12]
15	[5, 14, 2], [11, 2, 10]
17	[15, 0, 8], [1, 12, 12], [9, 12, 8]
19	[1, 6, 18], [17, 6, 6], [15, 6, 10]

Open Problem 1: Prove elementary that primitive solutions of $\ell^2 = a^2 + b^2 + c^2$, always exist ($\gcd(a, b, c) = 1$) for every ℓ odd.

Proposition *Suppose that $u' = (a, b, c)$ satisfies*

$$n_1 a + n_2 b + n_3 c = 0,$$

where $n_1^2 + n_2^2 + n_3^2 = \ell^2$ with all variables involved being integers. Then there exist $v = (a', b', c')$ such that $\ell u'$ and v define a lattice square in the plane of normal $n = (n_1, n_2, n_3)$.

Parametrization

Theorem: *Every primitive solution of* $\ell^2 = n_1^2 + n_2^2 + n_3^2$ *is, up to a permutation, given by*

$$n_1 = |2zy - 2tx|,\ n_2 = |2tz + 2yx|,\ n_3 = |z^2 - t^2 + x^2 - y^2|,$$

and $\ell = x^2 + y^2 + z^2 + t^2$, *for some integers* x, y, z *and* t.

- Lagrange's four-square theorem

- Divisibility in Gaussian Integers

Family of squares

$$u := (2ty + 2zx, 2tz - 2yx, t^2 - z^2 - y^2 + x^2)$$

$$v := (2zy - 2tx, z^2 - t^2 + x^2 - y^2, 2tz + 2yx)$$

with normal vector

$$n = (-x^2 + t^2 - y^2 + z^2, -2(tx + zy), 2(ty - zx)),$$

$$|n| = x^2 + y^2 + z^2 + t^2$$

Theorem: *The set of all ℓ so that $\sqrt{\ell}$ is the side-lengths for an embedded square in $\mathbb{Z}^3$ is the set of positive integers which are sums of two squares.*

Problem: *Given a point $P = (x, y, z) \in \mathbb{Z}^3$ in the plane of equation $n_1 x + n_2 y + n_3 z = 0$ with $n_1^2 + n_2^2 + n_3^2 = \ell^2$ for some $\ell \in \mathbb{N}$, then the number $x^2 + y^2 + z^2$ is actually a sum of two squares.*

Example: $37x + 46y + 22z = 0$

Squares in $\mathbb{R}^4$

$$O_{a,b,c,d} = \begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{bmatrix}.$$

Theorem: Given integer vectors $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$ such that $0 < \ell = u_1^2 + u_2^2 + u_3^2 + u_4^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2$, $u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 = 0$,

then there exist an odd $k \in \mathbb{N}$ dividing ℓ and two vectors w_1 and w_2 with integer coordinates such that

$w_1 = (0, \alpha_1 - \beta_1, \alpha_2 - \beta_2, \alpha_3 - \beta_3)$ and

$w_2 = (\alpha_3 - \beta_3, -\alpha_2 - \beta_2, \alpha_1 + \beta_1, 0)$ with

$k^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \beta_1^2 + \beta_2^2 + \beta_3^2$,

where α_i, β_i are of the same parity, and $u, v \in \{w_1, w_2\}^\perp$. Moreover, we can permute the coordinates of u and v and/or change their signs in order to have w_1 and w_2 linearly independent.

Open Problem 2: Let us assume that $k \in \mathbb{N}$ is odd such that we have the two integer representations $k^2 = a^2+b^2+c^2 = a'^2+b'^2+c'^2$, such that $\gcd(a,b,c,a',b',c') = 1$, $c > c'$, $a \equiv a'$, $b \equiv b'$, and $c \equiv c' \pmod 2$. Given the two vectors

$$w_1 = (0, a-a', b-b', c-c') \text{ and } w_2 = (c-c', -b-b', a+a', 0)$$

then the lattice $\{w_1, w_2\}^{\perp} \cap \mathbb{Z}^4$ has a fundamental domain of "volume" equal to k.

Theorem: Given k odd, and two different representations

$k^2 = a^2+b^2+c^2 = a'^2+b'^2+c'^2$, with $\gcd(a,b,c,a',b',c') = 1$, $c' > c$,

and a, a' both odd. Then if we set $\Delta_{12} = \frac{a'-a}{2}$, $\Delta_{34} = \frac{a+a'}{2}$, $\Delta_{13} = -\frac{b'-b}{2}$, $\Delta_{24} = \frac{b+b'}{2}$, $\Delta_{14} = \frac{c+c'}{2}$, and $\Delta_{23} = \frac{c'-c}{2}$, then the two dimensional space $\mathcal{S}$ of all vectors $[u,v,w,t] \in \mathbb{Z}^4$, such that

$$\begin{cases} (0)u + \Delta_{34}v + \Delta_{24}w + \Delta_{23}t = 0 \\ \Delta_{23}u + \Delta_{13}v + \Delta_{12}w + (0)t = 0 \end{cases} \tag{1}$$

contains a family of lattice squares.

Family of squares in 4-dimensions

$u = \pm(-ta - zb + yc - xd, -tb + za - yd - xc, -tc - zd - ya + xb, -td + zc + yb + xa)$

$v = \pm(ax - by - cz - dt, ay + bx + ct - dz, az - bt + cx + dy, at + bz - cy + dx)$

Open Problem 3: This gives them all.

Quaternions

(I) $\mathbb{H}(\mathbb{R})$ is the free $\mathbb{R}$-module over the symbols i, j, and k, with 1 the multiplicative unit;

(II) $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$.

Open Problem 4: (I) We assume that q_1 and q_2 are not right-divisible by quaternions of the form $p = \alpha + \beta i$, $|p| > 1$, then the square above is minimal.

(II) Assuming the hypothesis of (I), the fundamental domain has "volume" equal to

$$V = \frac{|q_1|^2 |q_2|^2}{\text{gcd}(|q_1|^2, |q_2|^2)}.$$

The Ehrhart polynomial is

$$E_{\Box}(t) = gcd(|q_1|^2, |q_2|^2) t^2 + (D_1 + D_2)t + 1,$$

where $D_1 = \text{gcd}(u_1, u_2, u_3, u_4)$ and $D_2 = \text{gcd}(v_1, v_2, v_3, v_4)$.

Cubes in $\mathbb{R}^3$

$$C_\ell = \frac{1}{\ell}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \tag{2}$$

$$L(C_\ell, t) = \begin{matrix} \ell^3 t^3 + \ell(d_1 + d_2 + d_3)t^2 + (d_1 + d_2 + d_3)t + 1 \text{ or} \\ (\ell t + 1)[\ell^2 t^2 + (d_1 + d_2 + d_3 - \ell)t + 1] \end{matrix}, \; t \in \mathbb{N}. \tag{3}$$

Cubes in $\mathbb{R}^4$

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}, \tag{4}$$

$$L(C_\ell, t) = \ell D_4 t^3 + \Delta t^2 + \Delta' t + 1 \ \ t \in \mathbb{N}, \tag{5}$$

where $\Delta := \delta_{12} + \delta_{23} + \delta_{23}$ and $\Delta' := D_1 + D_2 + D_3$.

Hypercubes in $\mathbb{R}^4$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$4t^4 + 8t^3 + 8t^2 + 4t + 1 = (2t^2 + 2t + 1)^2$$

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{bmatrix}$$

$$9t^4 + 12t^3 + 6t^2 + 4t + 1 = (t+1)(3t+1)(3t^2+1)$$

Hadamard Matrix

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

$$16t^4 + 16t^3 + 12t^2 + 4t + 1 = (1 + 2t + 4t^2)^2$$

Cross product

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$25t^4 + 20t^3 + 14t^2 + 4t + 1 = (1 + 2t + 5t^2)^2$$

$$\frac{1}{\sqrt{6}}\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

$$36t^4+24t^3+8t^2+4t+1$$

$$\frac{1}{\sqrt{7}}\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & -2 & -1 & 1 \\ 1 & 1 & -2 & -1 \\ 1 & -1 & 1 & -2 \end{bmatrix}$$

$$49t^4+28t^3+6t^2+4t+1$$

Cross product

$$\frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 2 & -1 & -2 \\ 0 & 1 & -2 & 2 \end{bmatrix}$$

$$81t^4 + 54t^3 + 18t^2 + 6t + 1$$
$$= (3t+1)^2(9t^2+1)$$

$$\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 & 0 \\ 2 & -2 & 0 & 1 \\ 1 & 0 & -2 & -2 \\ 0 & 1 & -2 & 2 \end{bmatrix}$$

$$81t^4+36t^3+6t^2+4t+1$$

$$\frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & -2 & -1 & 1 \\ 1 & 1 & -2 & -2 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

$$100t^4+40t^3+16t^2+4t+1$$

Cross Product

$$\frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$100t^4 + 40t^3 + 24t^2 + 6t + 1 = (10t^2 + 2t + 1)^2$$

$$\frac{1}{\sqrt{11}}\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & -3 & 0 & 1 \\ 1 & 0 & -3 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix},$$

$$121t^4+44t^3+6t^2+4t+1$$

$$\frac{1}{\sqrt{13}}\begin{bmatrix} 2 & 2 & 2 & 1 \\ 2 & -2 & 1 & -2 \\ 2 & -1 & -2 & 2 \\ 1 & 2 & -2 & -2 \end{bmatrix}$$

$$169t^4+53t^3+6t^2+4t+1$$

Hypercubes

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \quad vs \quad \begin{pmatrix} -1 & 3 & 1 & 4 \\ 1 & 0 & 5 & -1 \\ 3 & -3 & 0 & 3 \\ -4 & -3 & 1 & 1 \end{pmatrix}$$

Open Question 5: Find a good description of all lattice hypercubes in $\mathbb{R}^4$.

Ehrhart polynomial for Hypercubes

$$E_{H(\ell)}(t) = \ell^2 t^4 + \alpha_1 t^3 + \alpha_2 t^2 + \alpha_3 t + 1$$

Proposition

♣ $\alpha_1 = \ell(D_1 + D_2 + D_3 + D_4)$

♣ There is a linear equation between α_2 and α_3 (yet to be determined)

Open Problem 6: $\alpha_3 = D_1 + D_2 + D_3 + D_4$

Open Question 7: What is happening in dimensions bigger than 4?

Open Question 8: Do octonions clarify things for dimensions $d \in \{5, 6, 7, 8\}$?

Thank you!