Ehrhart polynomial for lattice squares, cubes, and hypercubes

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Abstract

We are investigating the problem of constructing integer lattice squares, cubes and hypercubes in Euclidean spaces of dimension less then or equal to four. We then study the number of lattice points inside of regions with such boundaries. For tools, we are using some results from the theory of Ehrhart polynomials and elementary number theory. Some preliminary results are presented and fresh conjectures will be shared.
The Ehrhart’s polynomial, \( L(P,t) \) counts how many lattice points are inside \( tP \)

\[
L(P,t) = c_0 t^d + c_1 t^{d-1} + \ldots + c_{d-1} t + c_d
\]

- \( d, c_0, c_1 \) and \( c_d \)
- \((-1)^d L(P,-t)\) (combinatorial reciprocity theorem)
• \( L(P \times Q, t) = L(P, t)L(Q, t) \) (cross product of polytopes)

• **Exercise:** The “volume” of the fundamental domain of the sub-lattice \( \mathbb{Z}^d \) of all solutions \((x_1, x_2, \ldots x_d) \in \mathbb{Z}^d\) satisfying

\[
a_1x_1 + a_2x_2 + \ldots + a_dx_d = 0
\]

is equal to \( \sqrt{a_1^2 + a_2^2 + \ldots + a_d^2} \), where \( a_1, a_2, a_3, \ldots, a_d \in \mathbb{Z} \) and \( \text{gcd}(a_1, a_2, \ldots, a_d) = 1. \)


Sequence of almost perfect squares in two dimensions

0, 1, 4, 5, 9, 12, 13, 16, 17, 24, 25, 28, 33, 36, 37, 40, 41, 49, 52, 57, 60, 61, 64, 65, 72, 73, 81, 84, 85, 88, 96, 97, 100, ..., A194154

... ↗ 2015, ↘ 2016
Given $a, b$ with $\gcd(a, b) = 1$, then

$$E_\square(T) = (a^2 + b^2)t^2 + 2t + 1, \ t \in \mathbb{N}.$$ 

Problem 1: Find the number of integer solutions $(x, y)$ of the system

$$4x + 3y, 4y - 3x \in [1, 99].$$

Problem 2: Find the number of integer solutions $(x, y)$ of the system

$$4x + 5y, 4y - 5x \in [1, 81].$$
**Theorem:** Given $a$, $b$ with $\gcd(a, b) = 1$ and $t \in \mathbb{N}$, then

$$\#\{(x, y) \in \mathbb{Z}^2 : ax + by, ay - bx \in [1, t(a^2 + b^2) - 1]\} = (a^2 + b^2)t^2 - 2t + 1.$$
Dimension 3 (squares)

\[ u = (a, b, c), \quad v = (a', b', c'), \quad a, b, c, a', b', c' \in \mathbb{Z} \text{ such that} \]
\[ a^2 + b^2 + c^2 = a'^2 + b'^2 + c'^2 = \ell \text{ and } aa' + bb' + cc' = 0. \]

\[ \diamond \gcd(a, b, c, a', b', c') = 1 \]

\[ \diamond \quad d = \gcd(a, b, c), \quad d' = \gcd(a', b', c') \text{ and} \]

\[ \diamond \quad D = \gcd(bc' - b'c, ac' - a'c, ab' - b'a) \]
**Theorem:** The Ehrhart polynomial of a lattice square embedded into $\mathbb{R}^3$, described above and with the notation introduced is given by

$$E_{\square}(t) = D t^2 + (d + d')t + 1.$$  

**Proof:**
Examples

$\vartriangleleft \mathbf{u} = (3, -3, 0)$ and $\mathbf{v} = (1, 1, 4)$

$\vartriangleleft \mathbf{u} = 5(8, 12, 9)$ and $\mathbf{v} = 17(0, -3, 4)$
\[ \ell^2 = a^2 + b^2 + c^2, \ a, b, c, \ell \in \mathbb{Z} \]

<table>
<thead>
<tr>
<th>\ell</th>
<th>[a,b,c], \gcd(a, b, c) = 1, 0 &lt; a \leq b \leq c</th>
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<tr>
<td>1</td>
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<td>19</td>
<td>[1, 6, 18], [17, 6, 6], [15, 6, 10]</td>
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Open Problem 1: Prove elementary that primitive solutions of $\ell^2 = a^2 + b^2 + c^2$, always exist ($\gcd(a, b, c) = 1$) for every $\ell$ odd.
**Proposition** Suppose that \( u' = (a, b, c) \) satisfies
\[
n_1a + n_2b + n_3c = 0,
\]
where \( n_1^2 + n_2^2 + n_3^2 = \ell^2 \) with all variables involved being integers. Then there exist \( v = (a', b', c') \) such that \( \ell u' \) and \( v \) define a lattice square in the plane of normal \( n = (n_1, n_2, n_3) \).
Parametrization

**Theorem:** Every primitive solution of $\ell^2 = n_1^2 + n_2^2 + n_3^2$ is, up to a permutation, given by

\[
n_1 = |2zy - 2tx|, \quad n_2 = |2tz + 2yx|, \quad n_3 = |z^2 - t^2 + x^2 - y^2|,
\]

and $\ell = x^2 + y^2 + z^2 + t^2$, for some integers $x$, $y$, $z$ and $t$.

- Lagrange’s four-square theorem
- Divisibility in Gaussian Integers
Family of squares

\[ u := (2ty + 2zx, 2tz - 2yx, t^2 - z^2 - y^2 + x^2) \]

\[ v := (2zy - 2tx, z^2 - t^2 + x^2 - y^2, 2tz + 2yx) \]

with normal vector

\[ n = (-x^2 + t^2 - y^2 + z^2, -2(tx + zy), 2(ty - zx)), \]

\[ |n| = x^2 + y^2 + z^2 + t^2 \]
**Theorem:** The set of all $\ell$ so that $\sqrt{\ell}$ is the side-lengths for an embedded square in $\mathbb{Z}^3$ is the set of positive integers which are sums of two squares.

**Problem:** Given a point $P = (x, y, z) \in \mathbb{Z}^3$ in the plane of equation $n_1 x + n_2 y + n_3 z = 0$ with $n_1^2 + n_2^2 + n_3^2 = \ell^2$ for some $\ell \in \mathbb{N}$, then the number $x^2 + y^2 + z^2$ is actually a sum of two squares.

Example: $37x + 46y + 22z = 0$
Squares in $\mathbb{R}^4$

$$O_{a,b,c,d} = \begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{bmatrix}.$$
**Theorem:** Given integer vectors $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$ such that $0 < \ell = u_1^2 + u_2^2 + u_3^2 + u_4^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2$, $u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 = 0$,

then there exist an odd $k \in \mathbb{N}$ dividing $\ell$ and two vectors $w_1$ and $w_2$ with integer coordinates such that

$$w_1 = (0, \alpha_1 - \beta_1, \alpha_2 - \beta_2, \alpha_3 - \beta_3)$$

and

$$w_2 = (\alpha_3 - \beta_3, -\alpha_2 - \beta_2, \alpha_1 + \beta_1, 0)$$

with

$$k^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \beta_1^2 + \beta_2^2 + \beta_3^2,$$

where $\alpha_i, \beta_i$ are of the same parity, and $u, v \in \{w_1, w_2\}^\perp$. Moreover, we can permute the coordinates of $u$ and $v$ and/or change their signs in order to have $w_1$ and $w_2$ linearly independent.
Open Problem 2: Let us assume that $k \in \mathbb{N}$ is odd such that we have the two integer representations $k^2 = a^2 + b^2 + c^2 = a'^2 + b'^2 + c'^2$, such that $\gcd(a, b, c, a', b', c') = 1$, $c > c'$, $a \equiv a'$, $b \equiv b'$, and $c \equiv c' \pmod{2}$. Given the two vectors

$$w_1 = (0, a-a', b-b', c-c')$$
$$w_2 = (c-c', -b-b', a+a', 0)$$

then the lattice $\{w_1, w_2\}^\perp \cap \mathbb{Z}^4$ has a fundamental domain of "volume" equal to $k$. 
**Theorem:** Given $k$ odd, and two different representations

\[ k^2 = a^2 + b^2 + c^2 = a'^2 + b'^2 + c'^2, \]

with \( \gcd(a, b, c, a', b', c') = 1 \), \( c' > c \), and \( a, a' \) both odd. Then if we set \( \Delta_{12} = \frac{a' - a}{2} \), \( \Delta_{34} = \frac{a + a'}{2} \), \( \Delta_{13} = -\frac{b' - b}{2} \), \( \Delta_{24} = \frac{b + b'}{2} \), \( \Delta_{14} = \frac{c + c'}{2} \), and \( \Delta_{23} = \frac{c' - c}{2} \), then the two dimensional space \( S \) of all vectors \([u, v, w, t] \in \mathbb{Z}^4\), such that

\[
\begin{align*}
(0)u + \Delta_{34}v + \Delta_{24}w + \Delta_{23}t &= 0 \\
\Delta_{23}u + \Delta_{13}v + \Delta_{12}w + (0)t &= 0
\end{align*}
\]

contains a family of lattice squares.
Family of squares in 4-dimensions

\[ u = \pm (-ta - zb + yc - xd, -tb + za - yd - xc, -tc - zd - ya + xb, -td + zc + yb + xa) \]

\[ v = \pm (ax - by - cz - dt, ay + bx + ct - dz, az - bt + cx + dy, at + bz - cy + dx) \]

**Open Problem 3:** This gives them all.
**Quaternions**

(I) $H(\mathbb{R})$ is the free $\mathbb{R}$-module over the symbols $i$, $j$, and $k$, with 1 the multiplicative unit;

(II) $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. 
Open Problem 4: (I) We assume that \( q_1 \) and \( q_2 \) are not right-divisible by quaternions of the form \( p = \alpha + \beta i \), \(|p| > 1\), then the square above is minimal.

(II) Assuming the hypothesis of (I), the fundamental domain has “volume” equal to

\[
V = \frac{|q_1|^2 |q_2|^2}{\gcd(|q_1|^2, |q_2|^2)}.
\]

The Ehrhart polynomial is

\[
E_{\square}(t) = \gcd(|q_1|^2, |q_2|^2)t^2 + (D_1 + D_2)t + 1,
\]

where \( D_1 = \gcd(u_1, u_2, u_3, u_4) \) and \( D_2 = \gcd(v_1, v_2, v_3, v_4) \).
Cubes in $\mathbb{R}^3$

$$C_\ell = \frac{1}{\ell} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},$$ \hspace{1cm} (2)

$$L(C_\ell, t) = \ell^3 t^3 + \ell(d_1 + d_2 + d_3)t^2 + (d_1 + d_2 + d_3)t + 1 \text{ or } \hspace{1cm} (3)$$

$$(\ell t + 1)[\ell^2 t^2 + (d_1 + d_2 + d_3 - \ell)t + 1], \hspace{1cm} t \in \mathbb{N}.$$
Cubes in $\mathbb{R}^4$

\[
\begin{bmatrix}
    a_1 & b_1 & c_1 & d_1 \\
    a_2 & b_2 & c_2 & d_2 \\
    a_3 & b_3 & c_3 & d_3 \\
    a_4 & b_4 & c_4 & d_4
\end{bmatrix},
\]

(4)

\[L(C_\ell, t) = \ell D_4 t^3 + \Delta t^2 + \Delta' t + 1 \quad t \in \mathbb{N},\]

(5)

where $\Delta := \delta_{12} + \delta_{23} + \delta_{23}$ and $\Delta' := D_1 + D_2 + D_3$. 
Hypercubes in $\mathbb{R}^4$

\[
\begin{align*}
1 & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
\sqrt{2} & \begin{bmatrix} 4t^4 + 8t^3 + 8t^2 + 4t + 1 \\ = (2t^2 + 2t + 1)^2 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
1 & \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{bmatrix} \\
\sqrt{3} & \begin{bmatrix} 9t^4 + 12t^3 + 6t^2 + 4t + 1 \\ = (t + 1)(3t + 1)(3t^2 + 1) \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
1 & \begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \\
2 & \begin{bmatrix} 16t^4 + 16t^3 + 12t^2 + 4t + 1 \\ = (1 + 2t + 4t^2)^2 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
1 & \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \\
\sqrt{5} & \begin{bmatrix} 25t^4 + 20t^3 + 14t^2 + 4t + 1 \\ = (1 + 2t + 5t^2)^2 \end{bmatrix}
\end{align*}
\]
\[
\begin{align*}
\frac{1}{\sqrt{6}} & \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix} 36t^4 + 24t^3 + 8t^2 + 4t + 1 \\
\frac{1}{\sqrt{7}} & \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & -2 & -1 & 1 \\ 1 & 1 & -2 & -1 \\ 1 & -1 & 1 & -2 \end{bmatrix} 49t^4 + 28t^3 + 6t^2 + 4t + 1 \\
\end{align*}
\]

Cross product

\[
\frac{1}{3} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & -1 & -2 \\ 0 & 1 & -2 & 2 \end{bmatrix} 81t^4 + 54t^3 + 18t^2 + 6t + 1 = (3t + 1)^2(9t^2 + 1)
\]
\[
\frac{1}{\sqrt{10}} \begin{bmatrix}
2 & 2 & 1 & 1 \\
2 & -2 & -1 & 1 \\
1 & 1 & -2 & -2 \\
1 & -1 & 2 & -2
\end{bmatrix} \\
100t^4 + 40t^3 + 16t^2 + 4t + 1
\]

\[
\frac{1}{\sqrt{10}} \begin{bmatrix}
3 & -1 & 0 & 0 \\
1 & 3 & 0 & 0 \\
0 & 0 & 3 & 1 \\
0 & 0 & 1 & -3
\end{bmatrix}
\]

\[
100t^4 + 40t^3 + 24t^2 + 6t + 1 = (10t^2 + 2t + 1)^2
\]

\[
\frac{1}{\sqrt{11}} \begin{bmatrix}
3 & 1 & 1 & 0 \\
1 & -3 & 0 & 1 \\
1 & 0 & -3 & -1 \\
0 & 1 & -1 & 3
\end{bmatrix} \\
121t^4 + 44t^3 + 6t^2 + 4t + 1
\]

\[
\frac{1}{\sqrt{13}} \begin{bmatrix}
2 & 2 & 2 & 1 \\
2 & -2 & 1 & -2 \\
2 & -1 & -2 & 2 \\
1 & 2 & -2 & -2
\end{bmatrix} \\
169t^4 + 53t^3 + 6t^2 + 4t + 1
\]
Open Question 5: Find a good description of all lattice hypercubes in $\mathbb{R}^4$. 

$$\begin{pmatrix} 
a & -b & -c & -d 
b & a & -d & c 
c & d & a & -b 
d & -c & b & a \end{pmatrix} vs \begin{pmatrix} 
-1 & 3 & 1 & 4 
1 & 0 & 5 & -1 
3 & -3 & 0 & 3 
-4 & -3 & 1 & 1 \end{pmatrix}$$
Ehrhart polynomial for Hypercubes

\[ E_{H(\ell)}(t) = \ell^2 t^4 + \alpha_1 t^3 + \alpha_2 t^2 + \alpha_3 t + 1 \]

**Proposition**

\[ \alpha_1 = \ell(D_1 + D_2 + D_3 + D_4) \]

\[ \alpha_2 \text{ and } \alpha_3 \text{ (yet to be determined) } \]

**Open Problem 6:** \[ \alpha_3 = D_1 + D_2 + D_3 + D_4 \]
Open Question 7: What is happening in dimensions bigger than 4?

Open Question 8: Do octonions clarify things for dimensions $d \in \{5, 6, 7, 8\}$?
Thank you!