Fifth Annual Columbus State Calculus Contest

Sponsored by
The Columbus State University
Department of Mathematics
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The Columbus State University Mathematics faculty welcome you to this year's Pre-Calculus/Calculus contest. We wish you success on this test and in your future studies.

Instructions

This is a 120-minute, 20-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

The examination will be scored on the basis of $+23$ *for each correct answer,* -3 *for each incorrect selection, and* 0 *for each omitted item.*

No phones or any communication devices can be used. Calculators with CAS such as the TI-89 are not allowed. In fact, the test is designed in such a way that you do not really need a calculator. The problems denoted with $[\star^n]$ are tie-breaker problems, so more attention should be given to them. Possibly, include written justification for your answers, on the pages provided at the end of the test, especially for the tie-breaker problems. It is not necessary, but you may find useful reading the "Theoretical facts" part.

Do not open your test until instructed to do so!

Theoretical facts that you may find useful.

In this exam:

(1) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(2) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

(3) The derivative of a function f is denoted by f'. Second and third derivatives are denoted by f'' and f''' respectively.

Theorem 1: If a function, which is two times differentiable, satisfies $f''(x) \geq 0$ for x in some interval I, we can conclude that f is convex upward on I.

Theorem 2: A differentiable function for which $f'(x)$ changes sign at a, then f has a local extrema at a.

Theorem 3: The area of the region between the graphs of f and g (continuous functions), $x \in [a, b]$, is given by

$$A = \int_a^b |f(x) - g(x)| dx$$

Theorem 4: The fundamental theorem of calculus:

(a) If f is continuous on the interval $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is differentiable and $F'(x) = f(x)$ for all $x \in [a, b]$.

(b) If f Riemann integrable on $[a, b]$ and F is an antiderivative of f, then $\int_a^b f(t)dt = F(b) - F(a)$.

Theorem 5: If f is differentiable and $f'(x) \geq 0$ on some interval, then f is non-decreasing on that interval.

Definition: A function is called *Riemann integrable* on $[a, b]$ if the following limit exists and it is independent of the partition $\Delta = (x_0 = a, x_1, ..., x_{n-1}, x_n = b)$, and the points $c_i \in [x_{i-1}, x_i]$, $i = 1, 2, ..., n$:

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = L, \ where \ ||\Delta|| = max\{x_i - x_{i-1} | i = 1, 2, ..., n\}.$$

If this is the case, L is written as $\int_a^b f(t)dt$.

Differentiation Rules: $(fg)' = f'g + fg'$, $(f/g)' = (f'g - fg')/g^2$, $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$ for all $|x| < 1$, $\frac{d}{dx} x^\alpha = \alpha x^{\alpha - 1}$ for all $x > 0$, $(f \circ g)' = (f' \circ g)g'$, $(u^v)' = vu^{v-1}u' + u^v(\ln u)v'$, $\frac{d}{dx} \tan x = \sec^2 x$, $\frac{d}{dx} e^u = u'e^u$, $\frac{d}{dx}(\arctan)(x) = \frac{1}{1+x^2}$.

(MVT) Mean Value Theorem: If $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), $F(b) - F(a) = (b - a)F'(c)$ for some $c \in (a, b)$.

Calculus Problems

1. For two real values a and b the function

$$f(x) = \begin{cases} (x+1)(x+2) & \text{if } \ x > 0 \\ \\ a\sin x + b\cos x, & \text{if } \ x \leq 0, \end{cases}$$

is continuous and differentiable at 0. What is $a - b$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

2. The functions $F(x) = \sin x$ and $G(x) = \cos x$ are defined for every real number x. Cauchy's theorem applied to F and G on the interval $[a, b]$, $0 < a < b < \pi$, gives

$$\frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(c)}{G'(c)}$$

with $c \in (a, b)$. For $a = \frac{9}{10}$ and $b = \frac{31}{10}$, what is $2c - 1$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

3. The function

$$g(x) = \left(\frac{x^2}{2} + x + 1\right)\cosh(x)$$

is defined by this rule for every real number x. For every natural number n, $g^{(n)}$ denotes the n^{th} derivative of g. For how many values of n we have

$$g^{(n)}(0) = 2017 \ ?$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

4. Suppose that f is defined on $\mathbb{R}$ by the rule

$$f(x) = (1 - x)(1 + x^2).$$

The function is invertible and we denote its inverse by f^{-1}. If $h = f^{-1} \circ \ln \circ f$, or in other words

$$h(x) = f^{-1}(\ln(f(x))), \quad x < 1,$$

what is $3 + \frac{1}{h'(0)}$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

5. The function

$$G(x) = \frac{x+3}{(x^2+3)^2},$$

defined for all real values of x, has three distinct inflection points. One of them is at $x = 1$. What is the product of the other two values of x corresponding to the inflection points?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

6. The function $g(x) = \cos(x) + 3\sin(2x)$ defined on the whole real line, has a maximum value of $\frac{m}{n}\sqrt{5}$, where $\frac{m}{n}$ is a rational number written in reduced form. What is the value of $(m+n)/2$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

7. For m a positive integer, we have

$$\lim_{x\to\infty}\left[x^3 \ln\left(\frac{x+1}{x}\right) + \frac{x}{2} - x^2\right] = \frac{1}{m}.$$

What is m?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

8. The function h is defined for all real numbers and it is at least three times differentiable, satisfying

$$h'''(x) + h''(x) + h'(x) + h(x) = 0$$

for all x. Knowing that $h(0) = 5$, $h'(0) = 1$ and $h''(0) = -3$, what is $h^{(2017)}(0)$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

9. Consider the function $g(x) = \frac{-x}{x^2-4x+3}$ defined for all real numbers $x \in (1,3)$. For every natural number n, $g^{(n)}$ denotes the n^{th} derivative of g. Find

$$\frac{g^{(2017)}(2)}{2017!}.$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

10. The curve defined implicitly by

$$x^3 + y^3 = 3xy + 3$$

passes through the point $(2,1)$. The tangent line to the curve at the point $(2,1)$ intersects the curve at another point (a,b). What is $5a + 3b$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

11. The curve defined implicitly by

$$x^3 + y^3 = 3xy + 3$$

passes through the point $(2,1)$. Find $y''(2)/4$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

12. The equation of the tangent line to the graph of equation

$$(y+1)\ln(2x-3) - (x-5)\ln(3y-2) = 0$$

at the point $(2,1)$ passes through the point $(-7,\omega)$. What is ω?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

13. If

$$F(x) = \int_x^{4x} \frac{1}{3+(4-t)^2(\ln t)^2}dt$$

for $x > 0$, what is $F'(1)$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

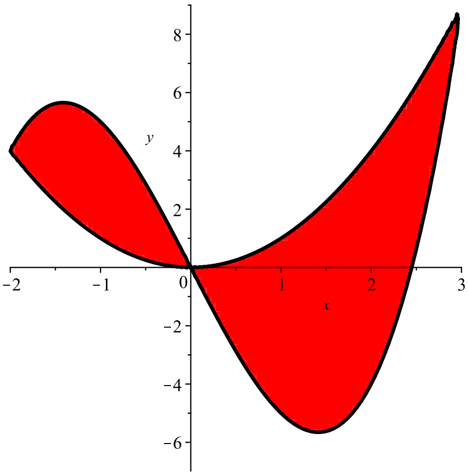
14. The graphs of $y = x^2$ and $y = x^3 - 6x$ for $x \in [-2, 3]$ are shown in the figure on the right. If A is the area between their graphs in this interval (shaded in red), then

$$A = \frac{m}{n}$$

is a rational number written in reduced form. What is $m - 21n$?

(A) 1 (B) 2 (C) 3

(D) 4 (E) 5



15. [$*^3$] We consider the quadratic function $G(x) = 2x(1-x)$ defined over the interval $I := [0, 1]$ with values in the interval I. For a positive integer n, we denote $G_n = \underbrace{G \circ G \circ \ldots \circ G}_{n\ times}$. Knowing that

$$\int_0^1 G_{2017}(x)dx = \frac{m}{n}$$

is in reduced terms, what is $n - 2m$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

16. The rational number $p = \frac{m}{n}$ (in reduced form) has the property that

$$\lim_{x \to \infty} x^p(\sqrt[3]{x+1} + \sqrt[3]{x-1} - 2\sqrt[3]{x})$$

is some non-zero real number. What is $m - n$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

17. The recurrent sequence $\{x_n\}$ satisfies the recurrence $x_{n+1} = \frac{6x_n}{1+x_n}$ for every $n \geq 1$ and $x_1 = 1/2017$. Knowing that $\{x_n\}$ is convergent to L, what is L?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

18. We define f by the rule $f(x) = 5(\sin x)^4 + 3(\cos x)^4$ for all real numbers x. Knowing that c is the smallest positive number with the property

$$f(c) = \frac{1}{\pi}\int_0^{\pi} f(x)dx$$

find $\frac{\pi}{c}$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

19. [$*^2$] We have for some natural number m

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2+k^2} = \frac{\ln m}{m}$$

Find m.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

20. [$*^1$] The function f is defined for $x \in [0, \frac{1}{e}]$ in the following way: for every $x \in [0, \frac{1}{e}]$, $f(x)$ is the solution of the equation

$$x = f(x)e^{-f(x)}.$$

Given that

$$\int_0^{1/e} f(x)dx = \frac{m}{e} - n,$$

where m and n are integers, what is m ?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5