**Sixth Annual Columbus State Calculus Contest**

Sponsored by
The Columbus State University
Department of Mathematics
March $3^{rd}$, 2018

************************

The Columbus State University Mathematics faculty welcome you to this year's Calculus contest. We wish you success on this test and in your future studies.

## Instructions

This is a 120-minute, 20-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response.

Throughout the exam, $\overline{AB}$ will denote the line segment from point $A$ to point $B$ and $AB$ will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

The examination will be scored on the basis of $+23$ for each correct answer, $-3$ for each incorrect choice and 0 for every omitted item. To get an expectation of 200 points, every test is awarded from the start with 156 points. The correct answer to each of the 20 questions results in a numerical value which may be 1, 2, 3, 4 or 5. If you get a value of 1, as your answer, you have to choose (A) on the scantron for that question. If you get a value of 2, as your answer, you have to choose (B) on the scantron for that question, and so on. The choices of A, B, C,... are equally distributed, i.e., there are four A's, four B's, and so on. Use this information as a strategy of your work through the test. The problems are arranged in the order of difficulty (from least difficult to most difficult) and so they are used as tie breaker in case of a tie.

No phones or any communication devices can be used. Calculators with CAS such as the TI-89 are not allowed. In fact, the test is designed in such a way that you do not really need a calculator. It is not necessary, but you may find useful reading the "Theoretical facts" part.

The answers to all 20 questions result in a numerical value which is 1, 2, 3, 4 or 5. If you get a value of 1, as your answer, you have to choose (A) on the scantron for that question. If you get a value of 2, as your answer, you have to choose (B) on the scantron for that question, and so on. The choices of A, B, C,... are equally distributed, i.e., there are four A's, four B's, and so on. Use this information as a strategy of your work through the test.

Do not open your test until instructed to do so!

**Theoretical facts that you may find useful.**

In this exam:

(1) Unless otherwise specified, the domain of a function $f$ is assumed to be the set of all real numbers $x$ for which $f(x)$ is a real number.

(2) The inverse of a trigonometric function $f$ may be indicated using the inverse function notation $f^{-1}$ or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

(3) The derivative of a function $f$ is denoted by $f'$. Second and third derivatives are denoted by $f''$ and $f'''$ respectively.

**Theorem 1:** If a function, which is two times differentiable, satisfies $f''(x) \geq 0$ for $x$ in some interval $I$, we can conclude that $f$ is convex upward on $I$.

**Theorem 2:** A differentiable function for which $f'(x)$ changes sign at $a$, then $f$ has a local extrema at $a$.

**Theorem 3:** The area of the region between the graphs of $f$ and $g$ (continuous functions), $x \in [a, b]$, is given by

$$A = \int_a^b |f(x) - g(x)| dx$$

**Theorem 4:** The fundamental theorem of calculus:

(a) If $f$ is continuous on the interval $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is differentiable and $F'(x) = f(x)$ for all $x \in [a, b]$.

(b) If $f$ Riemann integrable on $[a, b]$ and $F$ is an antiderivative of $f$, then $\int_a^b f(t)dt = F(b) - F(a)$.

**Theorem 5:** If $f$ is differentiable and $f'(x) \geq 0$ on some interval, then $f$ is non-decreasing on that interval.

**Definition:** A function is called *Riemann integrable* on $[a, b]$ if the following limit exists and it is independent of the partition $\Delta = (x_0 = a, x_1, ..., x_{n-1}, x_n = b)$, and the points $c_i \in [x_{i-1}, x_i]$, $i = 1, 2, ..., n$:

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = L, \ where \ ||\Delta|| = max\{x_i - x_{i-1} | i = 1, 2, ..., n\}.$$

If this is the case, $L$ is written as $\int_a^b f(t)dt$.

**Differentiation Rules:** $(fg)' = f'g + fg'$, $(f/g)' = (f'g - fg')/g^2$, $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$ for all $|x| < 1$, $\frac{d}{dx} x^\alpha = \alpha x^{\alpha - 1}$ for all $x > 0$, $(f \circ g)' = (f' \circ g)g'$, $(u^v)' = vu^{v-1}u' + u^v(\ln u)v'$, $\frac{d}{dx} \tan x = \sec^2 x$, $\frac{d}{dx} e^u = u'e^u$, $\frac{d}{dx}(\arctan)(x) = \frac{1}{1+x^2}$.

**(MVT) Mean Value Theorem:** If $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on $(a, b)$, then $F(b) - F(a) = (b - a)F'(c)$ for some $c \in (a, b)$.

The natural exponential function is simply denoted by $x \to e^x$, $x \in \mathbb{R}$. The derivative of a function $f$ at a point $x = a$ is denoted by $f'(a)$ or $\frac{df}{dx}(a)$.

The notation $n \in \mathbb{N}$, for instance, means $n = 1, 2, 3, ....$

## Calculus Contest Problems

1. Let $f$ and $g$ be real valued differentiable functions such that

$$f(x) = (3x+1) \cdot [2 - xg(x)],$$

for all real numbers $x$. If $g(0) = 1$, find the value of $f'(0)$.

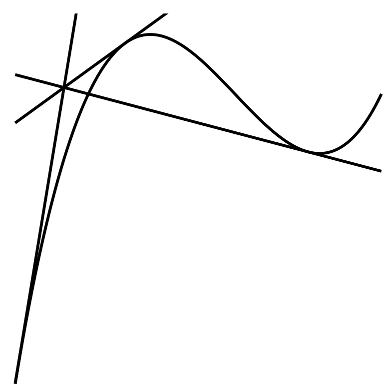
2. The function $P$ given by the rule $P(x) = (x^2 + 3x - 2)e^{-x}$ has two inflection points, one at $x = a$ and the other at $x = b$. What is $a + b$ ?

3. The functions $F(x) = x + \frac{1}{x}$ and $G(x) = x^3 - 3x$ are defined by these rules for every $x > 0$. Cauchy's theorem applied to $F$ and $G$ on the interval $[3, 11]$, gives

$$\frac{F(11) - F(3)}{G(11) - G(3)} = \frac{F'(c)}{G'(c)}$$

with $c \in (3, 11)$. What is $\frac{c^2}{11}$?

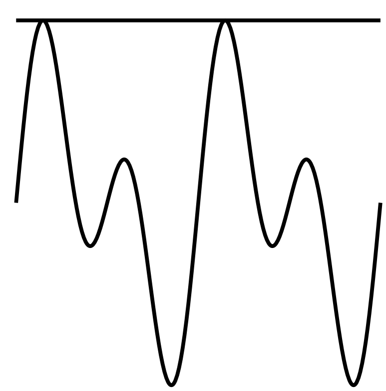
4. For three values of the parameter $m$, the line $y = m(x + \frac{7}{3}) + \frac{1}{3}$ is tangent to the curve $y = x^3 - 4x$ (see the adjacent figure). If $P$ is the product of all $x$ coordinates of these tangency points, find $\frac{2}{3}P$.



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

5. The function $g(x) = 14\sin(x) + 15\sin(2x)$ defined on the whole real line, has a maximum value of $\frac{m}{n}$, where $\frac{m}{n}$ is a rational number written in reduced form. What is the value of $m - n^3$ ?



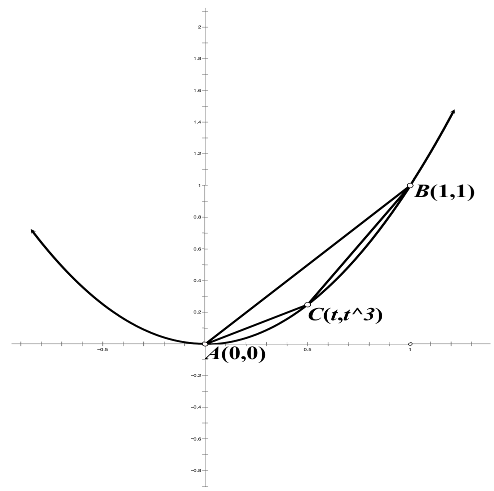
(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

6. Consider two fixed points $A = (0,0)$ and $B = (1,1)$ on the graph of $y = x^3$. We let $C_0 = (\alpha, \alpha^3)$ the point on this graph for which the area $S$ of the triangle $\triangle ABC_0$ is the biggest of all the triangles $\triangle ABC$ obtained with $C = (t, t^3)$ and $t$ in the interval $[0,1]$. If $S = \frac{\sqrt{3}}{n}$ where $n \in \mathbb{N}$, what is $n - 4$ ?

   (A) 1 (B) 2 (C) 3

   (D) 4 (E) 5




7. We let $y = x^{(e^x)}$ for all $x > 0$. Calculate

$$2 \lim_{x \to 0^+} \frac{dy}{dx}.$$

8. We denote by $L$ be the limit

$$\lim_{n \to \infty} n^2 \sum_{k=1}^{n} \frac{k}{(n^2 + k^2)^2}.$$

   Find $L^{-1}$.

9. Consider the sequence $\{x_n\}_{n \geq 1}$ defined by the formula

$$x_n = \sin \frac{2}{n^2} + \sin \frac{4}{n^2} + \sin \frac{6}{n^2} + \cdots + \sin \frac{2n}{n^2}, \quad n \in \mathbb{N}.$$

   Find the value of the limit $\lim_{n \to \infty} x_n$.

10. Consider the function $y = y(x)$ defined by

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{\ldots}}}}, \quad x > 0.$$

   Calculate $\frac{dx}{dy}(2)$.

11. The sequence of real numbers $\{x_n\}$ satisfies (the recurrence)

$$x_{n+1} = \frac{x_n + x_{n-1}}{2}, \quad n \geq 1,$$

   with $x_0 = 1$ and $x_1 = 7$. What is the $\lim_{n \to \infty} x_n$ ?

12. Knowing that

$$\int_0^{\ln 3} \frac{1}{5+2e^x+2e^{-x}}dx = \frac{1}{3}\ln(\epsilon),$$

where $\epsilon = \frac{a}{b}$ is a rational number in the reduced form, what is $a-b$ ?

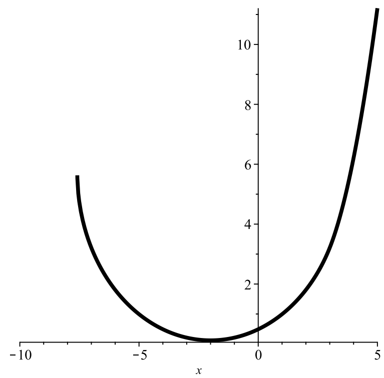
13. Find the value of the limit

$$\lim_{x\to 0} \frac{e^{\arctan x} - e^{\arcsin x}}{\arctan x - \arcsin x}.$$

14. For two real values $a$ and $b$ and the positive number $r$, the function

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 1 \\ b - \sqrt{r^2-(x+a)^2}, & \text{if } -7 \leq x \leq 1, \end{cases}$$

is continuous and two times differentiable at $x = 1$. What is $a$ ?




(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

15. The function $g$ is twice continuous differentiable on the real line and in addition

$$g(x^2-x) - (x^3-x^2-2x-1)g(x) = 2x+2$$

for all $x$. Calculate

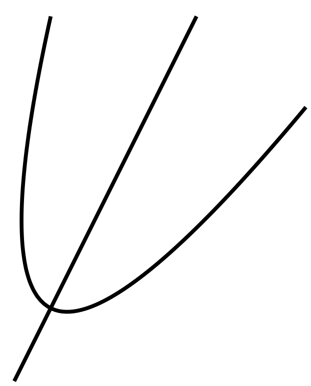
$$\int_0^1 xg''(2x)dx$$

.

16. The equation

$$(y-2x)^2 + \left(\frac{3}{5}\right)^2 = y - x$$

is a parabola with the vertex at $V(\alpha, \beta)$. What is $\frac{\beta}{\alpha}$?



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

17. The function $h$ is twice differentiable and satisfies the differential equation

$$h''(x) = \frac{h'(x)+h(x)}{2}, \text{ for all } x \in \mathbb{R}.$$

Knowing that $h(0) = 0$ and $h'(0) = 3$ find the limit

$$\lim_{x \to \infty} \frac{h(x)}{e^x}.$$

18. The continuous function $f$ satisfies

$$\int_0^x (x - t + 1) f(t) dt = 4x + \frac{x^3}{3} \text{ for all } x \in \mathbb{R}.$$

Find $f(0)$.

19. The function

$$g(x) = e^x \sin(x)$$

is defined by this rule for every real number $x$. For what positive value of $n$, the next equality holds true

$$g^{(2018)}(0) = 2^{25n^2 + (9n+1)^2} \text{ ?}$$

20. Calculate the value of the series

$$2^3 \frac{\pi^2}{4!} - 2^5 \frac{\pi^4}{6!} + ... = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n+1} \pi^{2n}}{(2n+2)!}.$$