Ordinary Differential Equations-Lecture Notes

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Preface

These lecture notes were written during the two semesters I have taught at the Georgia Institute of Technology, Atlanta, GA between the fall of 2005 and spring of 2006. I have used the well-known book by Edwards and Penny [5]. Some additional proofs are introduced in order to make the presentation as comprehensible as possible. Even that the audience was mostly engineering major students I have tried to teach this course for mathematics majors.

I have used the book of F. Diacu [4] when I taught the Ordinary Differential Equation class at Columbus State University, Columbus, GA in the Spring of 2005. This work determined me to have a closer interest in this area of mathematics and it influenced a lot my teaching style.

Chapter 1

Solving various types of differential equations

1.1 Lecture I

Quotation: *"The mind once expanded to the dimensions of larger ideas, never returns to its original size." Oliver Wendell Holmes*

Notions, concepts, definitions, and theorems: *Definition of a differential equations, the definition of a classical solution of a differential equation, classification of differential equations, an example of a real-world problem modeled by a differential equation, definition of an initial value problem.*

If we would like to start with some examples of differential equations, before we give a formal definition, let us think in terms of the main classes of functions that we studied in Calculus such as polynomial, rational, power functions, exponential, logarithmic, trigonometric, and inverse of trigonometric functions, what will be some equations that will be satisfied by these classes of functions or at least some of these types of functions?

For polynomials, we can think of a differential equation of the type:

$$\frac{d^n y}{dx^n}(x) = 0 \ \text{ for all } x \text{ in some interval,} \tag{1.1}$$

(with $n \in \mathbb{N}$) whose "solutions" would obviously include any arbitrary polynomial function y of x with degree at most $n-1$. In other words $y(x) = a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$ is a polynomial function that satisfies (1.1). Let us notice that there are n constants that we can choose as we like in the expression of y.

Let us say we consider a power function whose rule is given by $y(x) = x^\alpha$ with $\alpha \in \mathbb{R}$. Then by taking its derivative we get $\frac{dy}{dx}(x) = \alpha x^{\alpha-1}$, we see that we can make up a differential equation, in terms of only the function itself, that this function will satisfy

$$\frac{dy}{dx}(x) = \frac{\alpha y(x)}{x}, \quad \text{for } x \text{ in some interval contained in } (0, \infty). \tag{1.2}$$

For a rational function, lets say $y(x) = \dfrac{x+1}{2x+1}$, $x \in \mathbb{R} \setminus \{-\frac{1}{2}\}$, if we take the derivative of $y(x)$, we get $\dfrac{dy}{dx}(x) = -\dfrac{1}{(2x+1)^2}$ and since $y(x) = \dfrac{1}{2} + \dfrac{1}{2(2x+1)}$ a relatively natural way to involve the derivative and the function will be:

$$\frac{dy}{dx}(x) = -(2y(x) - 1)^2. \tag{1.3}$$

For a general rational function, it is not going to be that easy to find a corresponding differential equation that will be similar to (1.3), in which the variable x doesn't appear explicitly as in (1.2). These equations will be called later *autonomous differential equations*, as part of a wider class called *separable equations*. In such cases, most of the time the independent variable is dropped from the writing and so a differential equation as (1.3) can be rewritten simply as $y' = -(2y - 1)^2$.

Next, we are interested in finding a similar differential equation satisfied by an exponential function such as $y(x) = Ce^{kx}$, for some real constants C and k. It is easily seen that such a candidate can be:

$$\frac{dy}{dx}(x) = ky(x). \tag{1.4}$$

If we take $f(x) = \sin x$ and $g(x) = \cos x$ then we see that these two functions satisfy the following system of differential equations:

$$\begin{cases} \dfrac{df}{dx}(x) = g(x) \\ \\ \dfrac{dg}{dx}(x) = -f(x). \end{cases} \tag{1.5}$$

Let us observe that both functions satisfy the differential equation $f'' + f = 0$.

Now we are going to consider $f(x) = \arctan x$, $x \in \mathbb{R}$. Because the derivative of f is $f'(x) = \frac{1}{1+x^2}$ we can build a differential equation that f will satisfy:

$$f'(x) = \frac{1}{1 + (\tan f(x))^2} \quad \text{or} \quad f' = (1 + \tan^2 f)^{-1}. \tag{1.6}$$

Finally a function of two variables such as $f(x, y) = x^2 - y^2$, $x, y \in \mathbb{R}^2$ satisfies:

$$\frac{\partial f^2}{\partial x^2} + \frac{\partial f^2}{\partial y^2} = 0. \tag{1.7}$$

At this point we have enough examples and we will give a formal definition of a differential equation:

Definition 1.1.1. *A differential equation, shortly DE, is a relationship between a finite set of functions and their derivatives or partial derivatives of various order.*

Depending upon the domain of the functions involved, we have ordinary differential equations, or shortly ODE, when only one variable appears (as in equations (1.1)-(1.6)) or partial differential equations, shortly PDE, (as in (1.7)).

From the point of view of the number of functions involved we may have one function, in which case the equation is called *simple*, or we may have several functions, as in (1.5), in which case we say we have a *system* of differential equations.

Taking into account the structure of the equation we may have *linear differential equation* when the simple DE in question could be written in the form:

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + ... + a_n(x)y(x) = F(x), \tag{1.8}$$

or if we are dealing with a system of DE or PDE, each equation should be linear as before in all the unknown functions and their derivatives. In case such representations are not possible we are saying that the DE is *non-linear*. If the function F above is zero the linear equation is called *homogenous*. Otherwise, we are dealing with a non-homogeneous linear DE. If the differential equation does not contain (depend) explicitly on the independent variable or variables we call it an *autonomous* DE. As a consequence, the DE (1.2), is non-autonomous. As a result of these definitions the DE's (1.1), (1.2), (1.4), (1.5) and (1.7) are homogeneous linear differential equations.

The highest derivative that appears in a DE gives the *order*. For instance the equation (1.1) has order n and (1.7) has order two.

Definition 1.1.2. *We say that a function or a set of functions is/form a solution of a differential equation if the derivatives that appear in the DE exist on a certain domain and the DE is satisfied for all the values of the independent variables in that domain.*

This concept is usually called a classical solution of a differential equation. The domain for a DE is usually an interval or a union of intervals.

As an exercise, check that the function of two variables $F(x,t) = u(x+vt)+v(x-vt)$, where u and v are twice differentiable functions and v is some non-zero real number, is a solution of the 1-D wave equation:

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2}. \tag{1.9}$$

Next, we are going to deal with an example of DE that has rather a more real-world flavor than a theoretical one like the ones we have encountered so far.

Problem 1.1.3. [Calculus Textbook by Stewart] *We have a man (John) and his dog (Buddy) running on a straight beach (see Figure 7.1.1). At a given point in time, when the dog is* 12 *m from his owner, John starts running in the direction perpendicular to the beach with a certain constant speed. Buddy runs twice as fast and always toward John. The question is "where are they going to meet?"*

Solution: Let us assume that Buddy runs on a path given by the graph of a function f as in the figure above. Suppose that after a certain time, t, Buddy is at a position $(x, f(x))$ and John is on the y-axis at $(0, vt)$ where v is his speed in meters per second (assumed constant) of John. The fact that Buddy is running toward John at every time t, is going to give us a DE. This condition can be translated into the fact that the tangent line to the graph of f at $(x, f(x))$ passes through $(0, vt)$.

The equation of the tangent line is $Y - f(x) = f'(x)(X - x)$ and so the intersection with the y axis is $vt = f(x) - f'(x)x$. Let us assume the distance between Buddy and John is originally a ($a = 12$ in this problem). Buddy is running the distance $\int_x^a \sqrt{1 + f'(s)^2}ds$ which is supposed to be twice as big as vt (Buddy's speed is given to be twice as big v). Hence we get the equation $\int_x^a \sqrt{1 + f'(s)^2}ds = 2(f(x)-f'(x)x)$ in x, for every x in the interval $(0, a)$. By the Fundamental Theorem of Calculus, differentiating with respect to x we obtain: $-\sqrt{1 + f'(x)^2} = 2(-xf''(x))$ or

$$\frac{f''(x)}{\sqrt{1 + f'(x)^2}} = \frac{1}{2x}, x > 0.$$

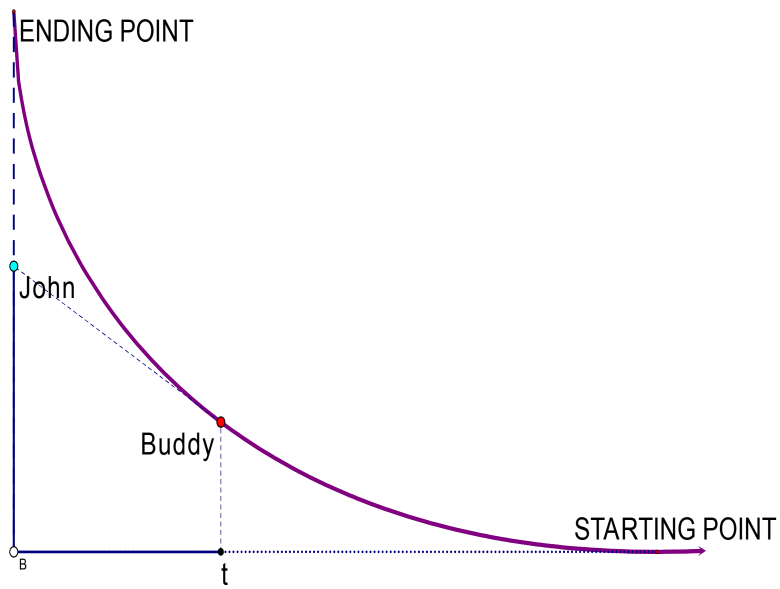


Figure 1.1: The man and his dog's trajectory

Integrating with respect to x gives

$$\ln(f'(x) + \sqrt{1 + f'(x)^2}) = \ln k\sqrt{x},$$

for some constant $k > 0$. Since $f'(a) = 0$ we determine k right away to be $k = \frac{1}{\sqrt{a}}$. Solving for $f'(x)$ gives $f'(x) = (\frac{\sqrt{x}}{\sqrt{a}} - \frac{\sqrt{a}}{\sqrt{x}})/2$. Integrating again with respect to x we obtain $f(x) = \frac{x\sqrt{x}}{3\sqrt{a}} - \sqrt{ax} + C$ for another constant C. Since $f(a) = 0$ we get $C = 2a/3$. Therefore $f(0) = \frac{2a}{3}$. So, the dog and its owner are going to meet at 8 meters from the point where John was when the "race" began. ■

In general, we like to know whether or not, of course under certain circumstances, a DE has a unique solution so that we may talk about *the* solution of the DE. This thing may happen but in the the general situation, this is hardly the case without some extra conditions such as initial conditions. To accomplish such a thing we usually consider the so-called *initial value problem* which takes the following form when we are dealing with a single, first-order ODE:

$$\begin{cases} \frac{dy}{dx}(x) = f(x, y(x)), x \in I, \\ \\ y(x_0) = y_0, x_0 \in I, y_0 \in J, I \times J \subset Domain(f), \end{cases} \tag{1.10}$$

where I and J are open intervals. For a system of ODE or a higher order ODE the initial value problem associated to it takes a slightly different form. We are going to see those at the appropriate time.

Homework: Problems 1-12, 27-31, 34, 37-43, 47 and 48, pages 8-9.

More challenging problems:

(a) Show that the initial value problem equation $f'' + f = 0$, $f(0) = f'(0) = 0$ has only the trivial solution $f \equiv 0$.

(b) Show that the equation $f''+f = 0$ has only the solution $f(x) = C_1 \sin x + C_2 \cos x$ for $x \in \mathbb{R}$, and some constants C_1 and C_2.

[**Putnam**A_3**,** 48^{th}**, 1987**] Let us consider the function $y = y(x)$ twice differentiable, satisfying $y''(x) - 2y'(x) + y(x) = 2e^x$ for all real x.

(i) If $y(x) > 0$ for all x, is it true that $y'(x) > 0$ for all x ? (include your arguments for the answer)

(ii) If $y'(x) > 0$ for all x, is it true that $y(x) > 0$ for all x ? (include your arguments for the answer)

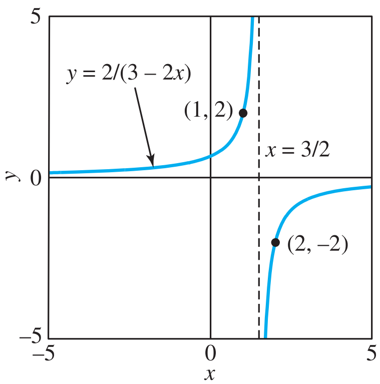


FIGURE 1.1.7. The solutions of $y' = y^2$ defined by $y(x) = 2/(3-2x)$.

so $2C - 2 = 1$, and hence $C = \frac{3}{2}$. With this value of C we obtain the desired solution

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$

Figure 1.1.7 shows the two branches of the graph $y = 2/(3 - 2x)$. The left-hand branch is the graph on $(-\infty, \frac{3}{2})$ of the solution of the given initial value problem $y' = y^2$, $y(1) = 2$. The right-hand branch passes through the point $(2, -2)$ and is therefore the graph on $(\frac{3}{2}, \infty)$ of the solution of the different initial value problem $y' = y^2$, $y(2) = -2$. ■

The central question of greatest immediate interest to us is this: If we are given a differential equation known to have a solution satisfying a given initial condition, how do we actually *find* or *compute* that solution? And, once found, what can we do with it? We will see that a relatively few simple techniques—separation of variables (Section 1.4), solution of linear equations (Section 1.5), elementary substitution methods (Section 1.6)—are enough to enable us to solve a variety of first-order equations having impressive applications.

1.1 Problems

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

1. $y' = 3x^2$; $y = x^3 + 7$
2. $y' + 2y = 0$; $y = 3e^{-2x}$
3. $y'' + 4y = 0$; $y_1 = \cos 2x$, $y_2 = \sin 2x$
4. $y'' = 9y$; $y_1 = e^{3x}$, $y_2 = e^{-3x}$
5. $y' = y + 2e^{-x}$; $y = e^x - e^{-x}$
6. $y'' + 4y' + 4y = 0$; $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$
7. $y'' - 2y' + 2y = 0$; $y_1 = e^x \cos x$, $y_2 = e^x \sin x$
8. $y'' + y = 3\cos 2x$, $y_1 = \cos x - \cos 2x$, $y_2 = \sin x - \cos 2x$
9. $y' + 2xy^2 = 0$; $y = \dfrac{1}{1 + x^2}$
10. $x^2y'' + xy' - y = \ln x$; $y_1 = x - \ln x$, $y_2 = \dfrac{1}{x} - \ln x$
11. $x^2y'' + 5xy' + 4y = 0$; $y_1 = \dfrac{1}{x^2}$, $y_2 = \dfrac{\ln x}{x^2}$
12. $x^2y'' - xy' + 2y = 0$; $y_1 = x\cos(\ln x)$, $y_2 = x\sin(\ln x)$

In Problems 13 through 16, substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant r for which $y = e^{rx}$ is a solution of the equation.

13. $3y' = 2y$
14. $4y'' = y$
15. $y'' + y' - 2y = 0$
16. $3y'' + 3y' - 4y = 0$

In Problems 17 through 26, first verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

17. $y' + y = 0$; $y(x) = Ce^{-x}$, $y(0) = 2$
18. $y' = 2y$; $y(x) = Ce^{2x}$, $y(0) = 3$
19. $y' = y + 1$; $y(x) = Ce^x - 1$, $y(0) = 5$
20. $y' = x - y$; $y(x) = Ce^{-x} + x - 1$, $y(0) = 10$
21. $y' + 3x^2y = 0$; $y(x) = Ce^{-x^3}$, $y(0) = 7$
22. $e^y y' = 1$; $y(x) = \ln(x + C)$, $y(0) = 0$
23. $x\dfrac{dy}{dx} + 3y = 2x^5$; $y(x) = \frac{1}{4}x^5 + Cx^{-3}$, $y(2) = 1$
24. $xy' - 3y = x^3$; $y(x) = x^3(C + \ln x)$, $y(1) = 17$
25. $y' = 3x^2(y^2 + 1)$; $y(x) = \tan(x^3 + C)$, $y(0) = 1$
26. $y' + y\tan x = \cos x$; $y(x) = (x + C)\cos x$, $y(\pi) = 0$

In Problems 27 through 31, a function $y = g(x)$ is described by some geometric property of its graph. Write a differential equation of the form $dy/dx = f(x, y)$ having the function g as its solution (or as one of its solutions).

27. The slope of the graph of g at the point (x, y) is the sum of x and y.
28. The line tangent to the graph of g at the point (x, y) intersects the x-axis at the point $(x/2, 0)$.
29. Every straight line normal to the graph of g passes through the point $(0, 1)$. Can you *guess* what the graph of such a function g might look like?
30. The graph of g is normal to every curve of the form $y = x^2 + k$ (k is a constant) where they meet.
31. The line tangent to the graph of g at (x, y) passes through the point $(-y, x)$.

In Problems 32 through 36, write—in the manner of Eqs. (3) through (6) of this section—a differential equation that is a mathematical model of the situation described.

32. The time rate of change of a population P is proportional to the square root of P.
33. The time rate of change of the velocity v of a coasting motorboat is proportional to the square of v.
34. The acceleration dv/dt of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.

35. In a city having a fixed population of P persons, the time rate of change of the number N of those persons who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor.

36. In a city with a fixed population of P persons, the time rate of change of the number N of those persons infected with a certain contagious disease is proportional to the product of the number who have the disease and the number who do not.

In Problems 37 through 42, determine by inspection at least one solution of the given differential equation. That is, use your knowledge of derivatives to make an intelligent guess. Then test your hypothesis.

37. $y'' = 0$ **38.** $y' = y$

39. $xy' + y = 3x^2$ **40.** $(y')^2 + y^2 = 1$

41. $y' + y = e^x$ **42.** $y'' + y = 0$

Problems 43 through 46 concern the differential equation

$$\frac{dx}{dt} = kx^2,$$

where k is a constant.

43. (a) If k is a constant, show that a general (one-parameter) solution of the differential equation is given by $x(t) = 1/(C - kt)$, where C is an arbitrary constant.
(b) Determine by inspection a solution of the initial value problem $x' = kx^2$, $x(0) = 0$.

44. (a) Assume that k is positive, and then sketch graphs of solutions of $x' = kx^2$ with several typical positive values of $x(0)$.
(b) How would these solutions differ if the constant k were negative?

45. Suppose a population P of rodents satisfies the differential equation $dP/dt = kP^2$. Initially, there are $P(0) = 2$ rodents, and their number is increasing at the rate of $dP/dt = 1$ rodent per month when there are $P = 10$ rodents. Based on the result of Problem 43, how long will it take for this population to grow to a hundred rodents? To a thousand? What's happening here?

46. Suppose the velocity v of a motorboat coasting in water satisfies the differential equation $dv/dt = kv^2$. The initial speed of the motorboat is $v(0) = 10$ meters per second (m/s), and v is decreasing at the rate of 1 m/s^2 when $v = 5$ m/s. Based on the result of Problem 43, long does it take for the velocity of the boat to decrease to 1 m/s? To $\frac{1}{10}$ m/s? When does the boat come to a stop?

47. In Example 7 we saw that $y(x) = 1/(C - x)$ defines a one-parameter family of solutions of the differential equation $dy/dx = y^2$. **(a)** Determine a value of C so that $y(10) = 10$. **(b)** Is there a value of C such that $y(0) = 0$? Can you nevertheless find by inspection a solution of $dy/dx = y^2$ such that $y(0) = 0$? **(c)** Figure 1.1.8 shows typical graphs of solutions of the form $y(x) = 1/(C - x)$. Does it appear that these solution curves fill the entire xy-plane? Can you conclude that, given any point (a, b) in the plane, the differential equation $dy/dx = y^2$ has exactly one solution $y(x)$ satisfying the condition $y(a) = b$?

48. (a) Show that $y(x) = Cx^4$ defines a one-parameter family of differentiable solutions of the differential equation $xy' = 4y$ (Fig. 1.1.9). **(b)** Show that

$$y(x) = \begin{cases} -x^4 & \text{if } x < 0, \\ x^4 & \text{if } x \geqq 0 \end{cases}$$

defines a differentiable solution of $xy' = 4y$ for all x, but is not of the form $y(x) = Cx^4$. **(c)** Given any two real numbers a and b, explain why—in contrast to the situation in part (c) of Problem 47—there exist infinitely many differentiable solutions of $xy' = 4y$ that all satisfy the condition $y(a) = b$.

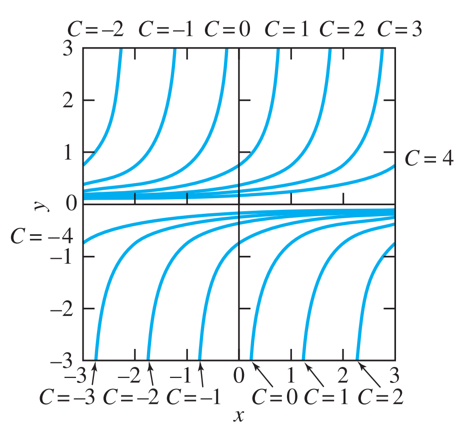


FIGURE 1.1.8. Graphs of solutions of the equation $dy/dx = y^2$.

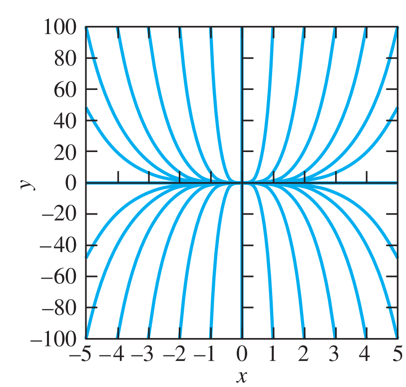


FIGURE 1.1.9. The graph $y = Cx^4$ for various values of C.

1.2 Lecture II

Quotation: *" An idea that can be used once is a trick. If it can be used more than once it becomes a method." George Polya and Gabor Szego*

Notions, concepts, definitions, and theorems: *Methods of study for differential equations, a/the general solution of a differential equation, particular solution, velocity and acceleration example, slope field and solution curves, existence theorem and an existence and uniqueness theorem.*

We say that differential equations are studied by *quantitative or exact methods* when they can be solved completely (i.e. all the solutions are known and could be written in closed form in terms of elementary functions or sometime special functions (or inverses of these type of functions). This reduces the study of DE to the study of functions of one or more real variables given in an explicit or implicit way.

As an example let us consider the equation in Exercise 4, page 16

$$\frac{dy}{dx} = \frac{1}{x^2}. \tag{1.11}$$

If we rewrite the equation as $\frac{d}{dx}(y(x)+\frac{1}{x})=0$ we see that we are dealing with a function whose derivative is zero. If we talk about solutions defined on an interval, the Mean Value Theorem from Calculus, tells us that $y(x)+\frac{1}{x}=C$ for some constant C and for all $x\in I$, I an interval not containing zero. Therefore any solution (as long as we consider the domains of solutions intervals like I) of the DE in (1.11) is of the form $y(x)=C-\frac{1}{x}$ for $x\in I$. So, we were able to solve the equation (1.11) exactly. To finish the Exercise 4, page 16, we determine C such that the initial value condition, $y(1)=5$, is satisfied too. This gives $C=6$ and $y(x)=\frac{6x-1}{x}$ for all $x\in I$.

There are also some other types of methods, called *analytical methods* or *qualitative methods* in which one can describe the behavior of a DE's solution such as existence, uniqueness, stability, chaotic or asymptotic character, boundlessness, periodicity, etc. without actually solving it exactly. This is an important and relatively new step in the theory of DE. Important because most of the differential equations cannot be solved exactly and are relatively new because they all started mainly at the end of the 19th century. One of the mathematicians who pioneered in this area was Henri Poincaré.

We can add to the list another type of method for studying DE to which are *numerical methods*. These methods mainly involve the use of a computer, a specially designed software following the procedure given by an approximation algorithm. In this part of mathematics one studies the algorithms and the error analysis involved in approximating the solution of a DE that in general cannot be studied with exact methods. Very good approximations could be obtained most of the time only locally (not too far from the initial value point).

Definition 1.2.1. *A general solution of a DE of order n is a solution that is given in terms of n independent parameters. A particular solution of a DE (relative to a general solution) is a solution that could be obtained from that general solution by simply choosing specific values of the parameters involved.*

If all the solutions of DE are particular solutions obtained from a general solution then this is referred to as *the* general solution.

As an example, we are going to show later that the general solution of the second order linear equation $y'' + 4y' + 4 = 0$ is $y(x) = (C_1 + C_2x)e^{-2x}$ for all $x \in I$.

Another example is the particular case of the movement of a body under the action of a constant force according to Newton's second law mechanics: $m\vec{a} = \vec{F}$. This implies that if we denote the position of the body relative to a fixed point in space by $x(t)$ (the dependent variable here being the time t, and units are fixed but not specified). Integrating twice the equation

$$\frac{d^2x}{dt^2}(t) = a, \tag{1.12}$$

we get

$$\boxed{x(t) = at^2/2 + v_0t + x_0,\ t \in \mathbb{R},} \tag{1.13}$$

where a is the constant acceleration, v_0 is the initial velocity and x_0 is the initial position. We can look at this as the general solution of the equation (1.12).

As an application let us work the following problem from the book (No. 36, page 17).

Problem 1.2.2. *If a woman has enough "spring" in her legs to jump vertically to a hight of 2.25 ft on the earth, how high could she jump on the moon, where the surface gravitational acceleration is (approximately) 5.3 $\frac{ft}{s^2}$?*

Solution: From the equation (1.13) we see that whatever her speed is initially, say v_0, on earth, she is going to get to a maximum height $h = v_0t - gt^2/2$ where t is given by the condition that $dx/dt = 0$ or $v_0 - gt = 0$. Hence, we get $h = v_0\frac{v_0}{g} - g(\frac{v_0}{g})^2/2$ or $h = \frac{v_0^2}{2g}$. (Notice that, so far, this is basically solving Problem 35, page 17). From this we can solve for v_0 and obtain $v_0 = \sqrt{2gh}$. On the moon she is going to use the same initial velocity (this is saying that the energy is the same). Hence $h_{max} = \frac{v_0^2}{2g_m} = \frac{2gh}{2g_m} = \frac{gh}{g_m}$ or $h_{max} = \frac{32\times 2.25}{5.3} = 13.58$ ft. ■

From now on in this Chapter we are going to concentrate on first order, single, ODE of the form:

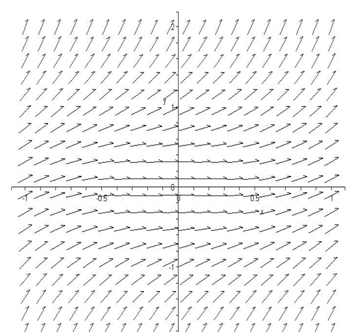
$$y' = f(x,y) \quad or \quad \frac{dy}{dx}(x) = f(x, y(x)). \tag{1.14}$$

We are trying to solve for y as a function of x. The best thing here is to look at an example. Let us take the example from the book, page 18, i.e. $y' = x^2 + y^2$ whose solution is not expressible in terms of simple functions. If we try Maple on this we get

$$y(x) = -x\frac{BesselJ(-3/4, x^2/2)C + BesselY(-3/4, x^2/2)}{BesselJ(1/4, x^2/2)C + BesselY(1/4, x^2/2)}.$$

We will learn later about Bessel functions which appear in the above expression of the general solution. This expression is useful if we want to do numerical calculations since Bessel functions can be expressed in terms of power series.

On the other hand if we imagine that at each point of coordinates (x, y) in the xy-plane we draw a little unit vector of slope $f(x, y) = x^2 + y^2$ then we get the picture below:



Vector / slope field

and we kind of see how the *solution curves* should look like. We are drawing next (of course, using a special tool like Maple) the solution curve passing through $(0, -1)$ for instance.

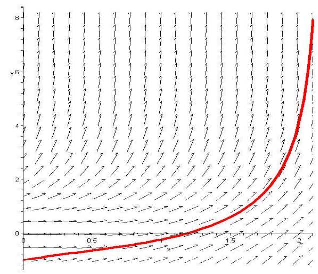


Figure 3

It seems like the vector field in Figure 2 defines uniquely the solutions curves. We are asking then the two fundamental questions in most of the mathematics when dealing with equations:

- When do we have at least a solution for (1.14)?
- If there exist a solution of (1.14) is that the only one?

The first problem is usually referred as *existence* problem and the second as the *uniqueness* problem. In general, in order to obtain existence for the DE (1.14) we only need continuity for the function f:

Theorem 1.2.3. (Peano) *If the function $f(x, y)$ is continuous on a rectangle $\mathcal{R} = \{(x, y)|a < x < b, c < y < d\}$, and if (x_0, y_0) in $\mathcal{R}$, then the initial value problem*

$$\begin{cases} \frac{dy}{dx}(x) = f(x, y(x)) \\ \\ y(x_0) = y_0, \end{cases} \tag{1.15}$$

has a solution in the neighborhood of x_0.

We need more than continuity in order to obtain uniqueness:

Theorem 1.2.4. (Cauchy) *Let $f(x, y)$ be continuous such that the derivative $\frac{\partial f}{\partial y}(x, y)$ exists and it is continuous on a rectangle $\mathcal{R} = \{(x, y)|a < x < b, c < y < d\}$, and if (x_0, y_0) in $\mathcal{R}$, then the initial value problem (1.15) has a solution which is unique on an interval around x_0.*

As an example we will look at Problem 30, page 28.

Problem 1.2.5. *Verify that if c is a constant, then the function defined piecewise by*

$$y(x) = \begin{cases} 1 \ \ \textit{if} \ \ x \le c \\ \cos(x-c) \ \ \textit{if} \ \ c < x < c+\pi \\ -1 \ \ \textit{if} \ \ x \ge c+\pi \end{cases} \tag{1.16}$$

satisfies the differential equation $y' = -\sqrt{1-y^2}$ for all $x \in \mathbb{R}$. Determine how many different solutions (in terms of a and b) the initial value problem

$$\begin{cases} y' = -\sqrt{1-y^2}, \\ y(a) = b \end{cases}$$

has.

Solution: It is not hard to see that the function y given in (8.1) is differentiable at each point and its derivative is actually

$$y'(x) = \begin{cases} 0 \ \ \text{if} \ \ x \le c \\ -\sin(x-c) \ \ \text{if} \ \ c < x < c+\pi \\ 0 \ \ \text{if} \ \ x \ge c+\pi \end{cases} \tag{1.17}$$

Hence if $x \le c$ or $x \ge c+\pi$ then the equation $y' = -\sqrt{1-y^2}$ is satisfied because $y' = -\sqrt{1-y^2} = 0$. If $c < x < c+\pi$ then $0 < x-c < \pi$ and then $\sin(x-c)$ is positive, which implies $\sqrt{1-\cos(x-c)^2} = \sin(x-c)$ and so the equation is satisfied in this case also.

For the second part of this problem, it is clear that if $|b| > 1$ we do not have any solution because $\sqrt{1-y(a)^2}$ is not a real number. If $b = 1$, we have infinitely many solutions, by just taking $c > a$, then the $y(x)$ defined by (8.1) is a solution of the initial value problem in the discussion. Similarly, we get infinitely many solutions if $b = -1$, in which case we have to take $c+\pi < a$ or $c < a-\pi$. If $-1 < b < 1$ we have a unique solution around the point a by Cauchy's Theorem but not on $\mathbb{R}$.

Homework:

Section 1.2 pages 15–17: 1-5, 11-15, 35 and 36;

Section 1.3 pages 26-27: 11-15, 27-33.

Suppose that a swimmer starts at the point $(-a, 0)$ on the west bank and swims due east (relative to the water) with constant speed v_S. As indicated in Fig. 1.2.5, his velocity vector (relative to the riverbed) has horizontal component v_S and vertical component v_R. Hence the swimmer's direction angle α is given by

$$\tan\alpha = \frac{v_R}{v_S}.$$

Because $\tan\alpha = dy/dx$, substitution using (18) gives the differential equation

$$\frac{dy}{dx} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right) \tag{19}$$

for the swimmer's trajectory $y = y(x)$ as he crosses the river.

Example 4 Suppose that the river is 1 mile wide and that its midstream velocity is $v_0 = 9$ mi/h. If the swimmer's velocity is $v_S = 3$ mi/h, then Eq. (19) takes the form

$$\frac{dy}{dx} = 3(1 - 4x^2).$$

Integration yields

$$y(x) = \int (3 - 12x^2)\,dx = 3x - 4x^3 + C$$

for the swimmer's trajectory. The initial condition $y\left(-\frac{1}{2}\right) = 0$ yields $C = 1$, so

$$y(x) = 3x - 4x^3 + 1.$$

Then

$$y\left(\tfrac{1}{2}\right) = 3\left(\tfrac{1}{2}\right) - 4\left(\tfrac{1}{2}\right)^3 + 1 = 2,$$

so the swimmer drifts 2 miles downstream while he swims 1 mile across the river. ■

1.2 Problems

In Problems 1 through 10, find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

1. $\dfrac{dy}{dx} = 2x + 1;\ y(0) = 3$

2. $\dfrac{dy}{dx} = (x-2)^2;\ y(2) = 1$

3. $\dfrac{dy}{dx} = \sqrt{x};\ y(4) = 0$

4. $\dfrac{dy}{dx} = \dfrac{1}{x^2};\ y(1) = 5$

5. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x+2}};\ y(2) = -1$

6. $\dfrac{dy}{dx} = x\sqrt{x^2+9};\ y(-4) = 0$

7. $\dfrac{dy}{dx} = \dfrac{10}{x^2+1};\ y(0) = 0$

8. $\dfrac{dy}{dx} = \cos 2x;\ y(0) = 1$

9. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}};\ y(0) = 0$

10. $\dfrac{dy}{dx} = xe^{-x};\ y(0) = 1$

In Problems 11 through 18, find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x_0 = x(0)$, and initial velocity $v_0 = v(0)$.

11. $a(t) = 50,\ v_0 = 10,\ x_0 = 20$

12. $a(t) = -20,\ v_0 = -15,\ x_0 = 5$

13. $a(t) = 3t,\ v_0 = 5,\ x_0 = 0$

14. $a(t) = 2t + 1,\ v_0 = -7,\ x_0 = 4$

15. $a(t) = 4(t+3)^2,\ v_0 = -1,\ x_0 = 1$

16. $a(t) = \dfrac{1}{\sqrt{t+4}},\ v_0 = -1,\ x_0 = 1$

17. $a(t) = \dfrac{1}{(t+1)^3},\ v_0 = 0,\ x_0 = 0$

18. $a(t) = 50\sin 5t,\ v_0 = -10,\ x_0 = 8$

In Problems 19 through 22, a particle starts at the origin and travels along the x-axis with the velocity function $v(t)$ whose graph is shown in Figs. 1.2.6 through 1.2.9. Sketch the graph of the resulting position function $x(t)$ for $0 \leqq t \leqq 10$.

19.

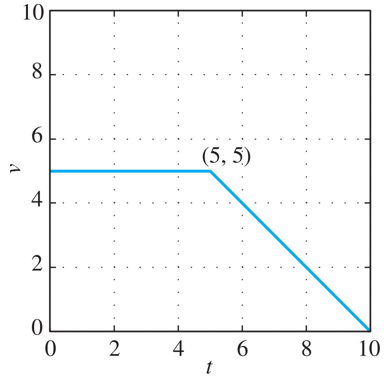


FIGURE 1.2.6. Graph of the velocity function $v(t)$ of Problem 19.

20.

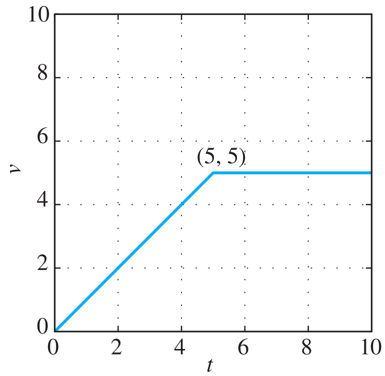


FIGURE 1.2.7. Graph of the velocity function $v(t)$ of Problem 20.

21.

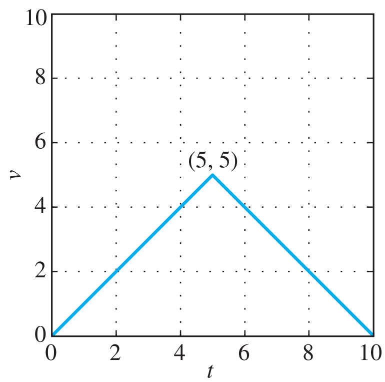


FIGURE 1.2.8. Graph of the velocity function $v(t)$ of Problem 21.

22.

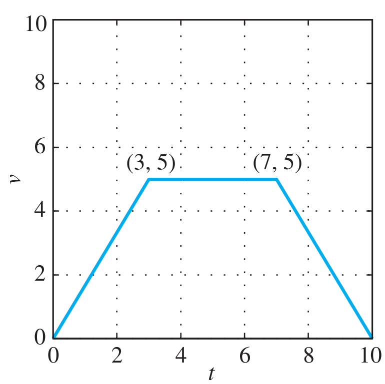


FIGURE 1.2.9. Graph of the velocity function $v(t)$ of Problem 22.

23. What is the maximum height attained by the arrow of part (b) of Example 3?

24. A ball is dropped from the top of a building 400 ft high. How long does it take to reach the ground? With what speed does the ball strike the ground?

25. The brakes of a car are applied when it is moving at 100 km/h and provide a constant deceleration of 10 meters per second per second (m/s^2). How far does the car travel before coming to a stop?

26. A projectile is fired straight upward with an initial velocity of 100 m/s from the top of a building 20 m high and falls to the ground at the base of the building. Find (a) its maximum height above the ground; (b) when it passes the top of the building; (c) its total time in the air.

27. A ball is thrown straight downward from the top of a tall building. The initial speed of the ball is 10 m/s. It strikes the ground with a speed of 60 m/s. How tall is the building?

28. A baseball is thrown straight downward with an initial speed of 40 ft/s from the top of the Washington Monument (555 ft high). How long does it take to reach the ground, and with what speed does the baseball strike the ground?

29. A diesel car gradually speeds up so that for the first 10 s its acceleration is given by

$$\frac{dv}{dt} = (0.12)t^2 + (0.6)t \quad (\text{ft/s}^2).$$

If the car starts from rest ($x_0 = 0$, $v_0 = 0$), find the distance it has traveled at the end of the first 10 s and its velocity at that time.

30. A car traveling at 60 mi/h (88 ft/s) skids 176 ft after its brakes are suddenly applied. Under the assumption that the braking system provides constant deceleration, what is that deceleration? For how long does the skid continue?

31. The skid marks made by an automobile indicated that its brakes were fully applied for a distance of 75 m before it came to a stop. The car in question is known to have a constant deceleration of 20 m/s^2 under these conditions. How fast—in km/h—was the car traveling when the brakes were first applied?

32. Suppose that a car skids 15 m if it is moving at 50 km/h when the brakes are applied. Assuming that the car has the same constant deceleration, how far will it skid if it is moving at 100 km/h when the brakes are applied?

33. On the planet Gzyx, a ball dropped from a height of 20 ft hits the ground in 2 s. If a ball is dropped from the top of a 200-ft-tall building on Gzyx, how long will it take to hit the ground? With what speed will it hit?

34. A person can throw a ball straight upward from the surface of the earth to a maximum height of 144 ft. How high could this person throw the ball on the planet Gzyx of Problem 33?

35. A stone is dropped from rest at an initial height h above the surface of the earth. Show that the speed with which it strikes the ground is $v = \sqrt{2gh}$.

36. Suppose a woman has enough "spring" in her legs to jump (on earth) from the ground to a height of 2.25 feet. If she jumps straight upward with the same initial velocity on the moon—where the surface gravitational acceleration is (approximately) 5.3 ft/s^2—how high above the surface will she rise?

37. At noon a car starts from rest at point A and proceeds at constant acceleration along a straight road toward point B. If the car reaches B at 12:50 P.M. with a velocity of 60 mi/h, what is the distance from A to B?

38. At noon a car starts from rest at point A and proceeds with constant acceleration along a straight road toward point C, 35 miles away. If the constantly accelerated car arrives at C with a velocity of 60 mi/h, at what time does it arrive at C?

39. If $a = 0.5$ mi and $v_0 = 9$ mi/h as in Example 4, what must the swimmer's speed v_S be in order that he drifts only 1 mile downstream as he crosses the river?

40. Suppose that $a = 0.5$ mi, $v_0 = 9$ mi/h, and $v_S = 3$ mi/h as in Example 4, but that the velocity of the river is given by the fourth-degree function

$$v_R = v_0\left(1 - \frac{x^4}{a^4}\right)$$

rather than the quadratic function in Eq. (18). Now find how far downstream the swimmer drifts as he crosses the river.

41. A bomb is dropped from a helicopter hovering at an altitude of 800 feet above the ground. From the ground directly beneath the helicopter, a projectile is fired straight upward toward the bomb, exactly 2 seconds after the bomb is released. With what initial velocity should the projectile be fired in order to hit the bomb at an altitude of exactly 400 feet?

42. A spacecraft is in free fall toward the surface of the moon at a speed of 1000 mph (mi/h). Its retrorockets, when fired, provide a constant deceleration of 20,000 mi/h^2. At what height above the lunar surface should the astronauts fire the retrorockets to insure a soft touchdown? (As in Example 2, ignore the moon's gravitational field.)

43. Arthur Clarke's *The Wind from the Sun* (1963) describes Diana, a spacecraft propelled by the solar wind. Its aluminized sail provides it with a constant acceleration of $0.001g = 0.0098$ m/s^2. Suppose this spacecraft starts from rest at time $t = 0$ and simultaneously fires a projectile (straight ahead in the same direction) that travels at one-tenth of the speed $c = 3 \times 10^8$ m/s of light. How long will it take the spacecraft to catch up with the projectile, and how far will it have traveled by then?

44. A driver involved in an accident claims he was going only 25 mph. When police tested his car, they found that when its brakes were applied at 25 mph, the car skidded only 45 feet before coming to a stop. But the driver's skid marks at the accident scene measured 210 feet. Assuming the same (constant) deceleration, determine the speed he was actually traveling just prior to the accident.

1.3 Slope Fields and Solution Curves

Consider a differential equation of the form

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

where the right-hand function $f(x, y)$ involves both the independent variable x and the dependent variable y. We might think of integrating both sides in (1) with respect to x, and hence write $y(x) = \int f(x, y(x))\,dx + C$. However, this approach does not lead to a solution of the differential equation, because the indicated integral involves the *unknown* function $y(x)$ itself, and therefore cannot be evaluated explicitly. Actually, there exists *no* straightforward procedure by which a general differential equation can be solved explicitly. Indeed, the solutions of such a simple-looking differential equation as $y' = x^2 + y^2$ cannot be expressed in terms of the ordinary elementary functions studied in calculus textbooks. Nevertheless, the graphical and numerical methods of this and later sections can be used to construct *approximate* solutions of differential equations that suffice for many practical purposes.

Slope Fields and Graphical Solutions

There is a simple geometric way to think about solutions of a given differential equation $y' = f(x, y)$. At each point (x, y) of the xy-plane, the value of $f(x, y)$ determines a slope $m = f(x, y)$. A solution of the differential equation is simply a differentiable function whose graph $y = y(x)$ has this "correct slope" at each

9. $\dfrac{dy}{dx} = x^2 - y - 2$

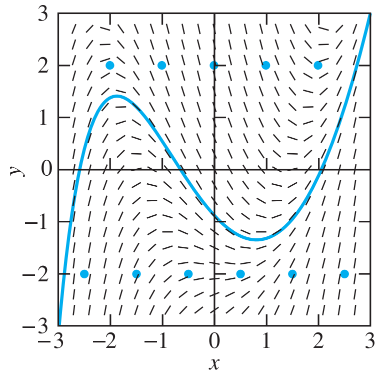


FIGURE 1.3.23.

10. $\dfrac{dy}{dx} = -x^2 + \sin y$

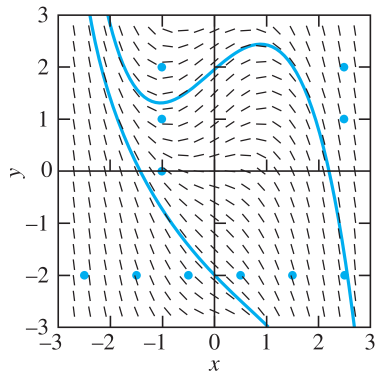


FIGURE 1.3.24.

A more detailed version of Theorem 1 says that, if the function $f(x, y)$ is continuous near the point (a, b), then at least one solution of the differential equation $y' = f(x, y)$ exists on some open interval I containing the point $x = a$ and, moreover, that if in addition the partial derivative $\partial f/\partial y$ is continuous near (a, b), then this solution is unique on some (perhaps smaller) interval J. In Problems 11 through 20, determine whether existence of at least one solution of the given initial value problem is thereby guaranteed and, if so, whether uniqueness of that solution is guaranteed.

11. $\dfrac{dy}{dx} = 2x^2y^2; \quad y(1) = -1$

12. $\dfrac{dy}{dx} = x \ln y; \quad y(1) = 1$

13. $\dfrac{dy}{dx} = \sqrt[3]{y}; \quad y(0) = 1$

14. $\dfrac{dy}{dx} = \sqrt[3]{y}; \quad y(0) = 0$

15. $\dfrac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 2$

16. $\dfrac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 1$

17. $y\dfrac{dy}{dx} = x - 1; \quad y(0) = 1$

18. $y\dfrac{dy}{dx} = x - 1; \quad y(1) = 0$

19. $\dfrac{dy}{dx} = \ln(1 + y^2); \quad y(0) = 0$

20. $\dfrac{dy}{dx} = x^2 - y^2; \quad y(0) = 1$

In Problems 21 and 22, first use the method of Example 2 to construct a slope field for the given differential equation. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution $y(x)$.

21. $y' = x + y, \quad y(0) = 0; \quad y(-4) = ?$

22. $y' = y - x, \quad y(4) = 0; \quad y(-4) = ?$

Problems 23 and 24 are like Problems 21 and 22, but now use a computer algebra system to plot and print out a slope field for the given differential equation. If you wish (and know how), you can check your manually sketched solution curve by plotting it with the computer.

23. $y' = x^2 + y^2 - 1, \quad y(0) = 0; \quad y(2) = ?$

24. $y' = x + \frac{1}{2}y^2, \quad y(-2) = 0; \quad y(2) = ?$

25. You bail out of the helicopter of Example 3 and pull the ripcord of your parachute. Now $k = 1.6$ in Eq. (3), so your downward velocity satisfies the initial value problem

$$\frac{dv}{dt} = 32 - 1.6v, \quad v(0) = 0.$$

In order to investigate your chances of survival, construct a slope field for this differential equation and sketch the appropriate solution curve. What will your limiting velocity be? Will a strategically located haystack do any good? How long will it take you to reach 95% of your limiting velocity?

26. Suppose the deer population $P(t)$ in a small forest satisfies the logistic equation

$$\frac{dP}{dt} = 0.0225P - 0.0003P^2.$$

Construct a slope field and appropriate solution curve to answer the following questions: If there are 25 deer at time $t = 0$ and t is measured in months, how long will it take the number of deer to double? What will be the limiting deer population?

The next seven problems illustrate the fact that, if the hypotheses of Theorem 1 are not satisfied, then the initial value problem $y' = f(x, y)$, $y(a) = b$ may have either no solutions, finitely many solutions, or infinitely many solutions.

27. **(a)** Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \leqq c, \\ (x - c)^2 & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 2\sqrt{y}$ for all x (including the point $x = c$). Construct a figure illustrating the fact that the initial value problem $y' = 2\sqrt{y}$, $y(0) = 0$ has infinitely many different solutions. **(b)** For what values of b does the initial value problem $y' = 2\sqrt{y}$, $y(0) = b$ have (i) no solution, (ii) a unique solution that is defined for all x?

28. Verify that if k is a constant, then the function $y(x) \equiv kx$ satisfies the differential equation $xy' = y$ for all x. Construct a slope field and several of these straight line solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem $xy' = y$, $y(a) = b$ has—one, none, or infinitely many.

29. Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \leqq c, \\ (x-c)^3 & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 3y^{2/3}$ for all x. Can you also use the "left half" of the cubic $y = (x-c)^3$ in piecing together a solution curve of the differential equation? (See Fig. 1.3.25.) Sketch a variety of such solution curves. Is there a point (a, b) of the xy-plane such that the initial value problem $y' = 3y^{2/3}$, $y(a) = b$ has either no solution or a unique solution that is defined for all x? Reconcile your answer with Theorem 1.

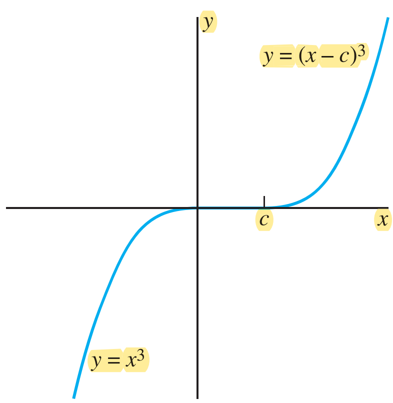


FIGURE 1.3.25. A suggestion for Problem 29.

30. Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} +1 & \text{if } x \leqq c, \\ \cos(x-c) & \text{if } c < x < c + \pi, \\ -1 & \text{if } x \geqq c + \pi \end{cases}$$

satisfies the differential equation $y' = -\sqrt{1-y^2}$ for all x. (Perhaps a preliminary sketch with $c = 0$ will be helpful.) Sketch a variety of such solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem $y' = -\sqrt{1-y^2}$, $y(a) = b$ has.

31. Carry out an investigation similar to that in Problem 30, except with the differential equation $y' = +\sqrt{1-y^2}$. Does it suffice simply to replace $\cos(x-c)$ with $\sin(x-c)$ in piecing together a solution that is defined for all x?

32. Verify that if $c > 0$, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{if } x^2 \leqq c, \\ (x^2-c)^2 & \text{if } x^2 > c \end{cases}$$

satisfies the differential equation $y' = 4x\sqrt{y}$ for all x. Sketch a variety of such solution curves for different values of c. Then determine (in terms of a and b) how many different solutions the initial value problem $y' = 4x\sqrt{y}$, $y(a) = b$ has.

33. If $c \neq 0$, verify that the function defined by $y(x) = x/(cx-1)$ (with the graph illustrated in Fig. 1.3.26) satisfies the differential equation $x^2y' + y^2 = 0$ if $x \neq 1/c$. Sketch a variety of such solution curves for different values of c. Also, note the constant-valued function $y(x) \equiv 0$ that does not result from any choice of the constant c. Finally, determine (in terms of a and b) how many different solutions the initial value problem $x^2y' + y^2 = 0$, $y(a) = b$ has.

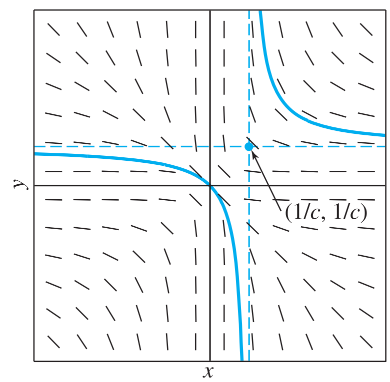


FIGURE 1.3.26. Slope field for $x^2y' + y^2 = 0$ and graph of a solution $y(x) = x/(cx-1)$.

34. (a) Use the direction field of Problem 5 to estimate the values at $x = 1$ of the two solutions of the differential equation $y' = y - x + 1$ with initial values $y(-1) = -1.2$ and $y(-1) = -0.8$.

(b) Use a computer algebra system to estimate the values at $x = 3$ of the two solutions of this differential equation with initial values $y(-3) = -3.01$ and $y(-3) = -2.99$.

The lesson of this problem is that small changes in initial conditions can make big differences in results.

35. (a) Use the direction field of Problem 6 to estimate the values at $x = 2$ of the two solutions of the differential equation $y' = x - y + 1$ with initial values $y(-3) = -0.2$ and $y(-3) = +0.2$.

(b) Use a computer algebra system to estimate the values at $x = 2$ of the two solutions of this differential equation with initial values $y(-3) = -0.5$ and $y(-3) = +0.5$.

The lesson of this problem is that big changes in initial conditions may make only small differences in results.

1.3 Lecture III

Quotation: *" Hardy, Godfrey H. (1877 - 1947) I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our "creations," are simply the notes of our observations. A Mathematician's Apology, London, Cambridge University Press, 1941. "*

Type of equations which can be solved with exact methods, notions, real-world applications: *Separable equations, implicit solution, singular solution, natural growth or decay equation, and general solution, Newton's law of cooling or heating and its general solution, Torricelli's law, liner first-order equations, and the general solution, mixture problems.*

One of the simplest cases in which the general solution could be found is the so-called *separable* differential equations. This is an equation of the form

$$y' = f(x)g(y) \tag{1.18}$$

where f and g are, let us say continuous functions on their domains that each contain an interval. Let us assume that g is not a constant. Then the function g is not zero for a set containing an interval too, say I. Then the equation (1.18) can be written equivalently as $\frac{y'}{g(y)} = f(x)$ if we assume that $y \in I$. We are going to treat the situation $g(y) = 0$ separately. Suppose $G(u)$ is an antiderivative of $\frac{1}{g(u)}$ on I, and F and antiderivative of f. Then the equation in question is equivalent to $\frac{d}{dx}(G(y(x)) - F(x)) = 0$ which means that the general solution should be

$$G(y(x)) - F(x) = C. \tag{1.19}$$

Most of the time, this equation cannot be solved in terms of $y(x)$ and we just say in that case that the solution, $y(x)$, is given implicitly.

The case $g(y_0) = 0$, will give solutions $y(x) = y_0$ which are usually called *singular solutions* unless (1.19) gives this solution for some value of the constant (parameter) C.

As an example let us take a look at Newton's law of cooling or heating: *the time rate of change of the temperature $T(t)$ of a body immersed in a medium of constant temperature M is proportional to the difference $M - T(t)$.*

This translates into

$$T'(t) = k(M - T(t)) \tag{1.20}$$

for some positive constant, which is a separable equation. Equivalently, this can be written as $\frac{T'(t)}{T(t)-M} = -k$ assuming that $T(t) \neq M$ at any time t. Integrating we obtain $\ln|T(t) - M| = -kt + C$ which implies $|T(t) - M| = e^{-kt}e^C$. If we make $t = 0$ we get that $e^C = \pm(T_0 - M)$ where T_0 is the initial temperature of the body. Then the expression of $T(t)$ becomes

$$\boxed{T(t) = M + (T_0 - M)e^{-kt}.} \tag{1.21}$$

Let us observe that the equation (1.20) admits only one other solution, namely the constant function $T(t) = M$, $t \in \mathbb{R}$, and that this solution is actually contained in (1.21) by simply taking $T_0 = M$. The equality above then is the general solution of (1.20) As an application of (1.21), let us take and solve Problem 43, page 42.

Problem 1.3.1. *A pitcher of buttermilk initially at* $25°$ *C is to be cooled by setting it on the front porch, where the temperature is* $0°$ *C. Suppose that the temperature of the buttermilk has dropped to* $15°$ *after* 20 *minutes. When will it be at* $5°$*?*

Solution: Using the formula (1.21), twice, we get $T(20) = 25e^{-20k} = 15$ which gives $k = \frac{1}{20}\ln(5/3)$ and so $T(t) = 25e^{-kt} = 5$. This last equation can then be solved for t to obtain $t = \frac{\ln 5}{k} = 20\frac{\ln 5}{\ln 5/3} \approx 63$ minutes. ■

Another application of separable DE is Torricelli's law: *suppose that a water tank has a hole with area a at its bottom and cross sectional area $A(y)$ for each height y, then the water flows in such a way the following DE is satisfied:*

$$A(y)\frac{dy}{dt} = -k\sqrt{y}. \tag{1.22}$$

where $k = a\sqrt{2g}$ and g is the the gravitational acceleration.

As an example of this situation let's take problem 62, page 43.

Problem 1.3.2. *Suppose that an initially full hemispherical water tank of radius* 1 *m has its flat side as its bottom. It has a bottom hole of radius* 1 *cm. If this bottom hole is opened at* 1 *P.M., when will the tank be empty?*

Solution: In the figure below we see that in order to calculate the cross-sectional area $A(y)$ corresponding to height y we need to apply the Pythagorean theorem: $A(y) = \pi(1 - y^2)$. Hence the equation that we get is $\pi\frac{1-y^2}{\sqrt{y}}\frac{dy}{dt} = -\pi\frac{1}{10000}\sqrt{2g}$.

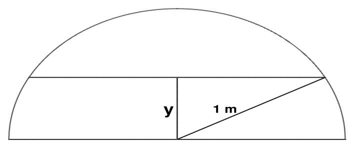


Figure 4

Integrating with respect to t we get $\frac{2}{5}y^{\frac{5}{2}} - 2y^{\frac{1}{2}} = \frac{t\sqrt{2g}}{10000} + C$. Since at $t = 0$ we have $y = 1$ the constant C is determined: $C = \frac{-8}{5}$. We are interested to see when is $y = 0$. This give $t = \frac{16000}{\sqrt{2g}} \approx = 3614$ second since g is measured here in m/s^2. This is approximately 1 hours so the tank will be empty around 2 P.M. (14 seconds after). ■

1.3.1 Linear First Order DE

These equation are equations of the type:

$$y' + P(x)y = Q(x), y(x_0) = y_0 \tag{1.23}$$

where P and Q are continuous on a given interval I ($x_0 \in I$). In order to solve (1.23), the trick is to multiply both side by $e^{R(x)}$ where $R(x)$ is an antiderivative of $P(x)$ on I. This way the equation becomes $\frac{dy}{dx}(y(x)e^{R(x)}) = Q(x)e^{R(x)}$ which after integration gives $ye^{R(x)} = \int Q(x)e^{R(x)}dx$. So the general solution of (1.23) is

$$y(x) = e^{-R(x)} \int Q(x)e^{R(x)}dx \tag{1.24}$$

Let us observe that if we have an initial value problem

$$\begin{cases} y' + P(x)y = Q(x), \\ y(x_0) = y_0 \end{cases} \tag{1.25}$$

where $x_0 \in I$, then we can take explicitly $R(x) = \int_{x_0}^{x} P(t)dt$ and (1.24) becomes

$$\boxed{y(x) = y_0e^{-R(x)} + \int_{x_0}^{x} Q(t)e^{R(t)-R(x)}dt, \quad x \in I.} \tag{1.26}$$

This proves the following theorem.

Theorem 1.3.3. *Given that P and Q are continuous functions on an interval I, the initial value problem (1.23) has a uniques solution on I given by (1.26).*

As an example let's solve the problem 12, page 54:

$$\begin{cases} xy' + 3y = 2x^5, \\ y(2) = 1. \end{cases} \tag{1.27}$$

The equation becomes $y' + \frac{3}{x}y' = 2x^4$. Since $R(x) = \int_1^x \frac{3}{t}dt = 3\ln x$ we obtain that we need to multiply the equation ($y' + \frac{3}{x}y' = 2x^4$) by $e^{R(x)} = x^3$. So, $x^3y' + 3x^2y = 2x^7$. The left hand side is $\frac{dy}{dx}(x^3y(x))$, so if we integrate from 2 to a we obtain $a^3y(a) - 8y(2) = \int_2^a 2x^7dx$. Equivalently, $a^3y(a) - 8 = 2(\frac{a^8}{8} - \frac{2^8}{8})$. So, the solution of this equation on $I = (0, \infty)$ is $y(a) = \dfrac{a^5}{4} - \dfrac{56}{a^3}$ for $a \in I$.

We are going to work out, as another application, the mixture problem 37 on page 54.

Problem 1.3.4. *A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well mixed brine in the tank flows at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?*

Solution: The tank is filling up at a speed of 2 gal/s and it is needed 300 gallons more to be full. So that is going to happen after 150 seconds. The volume of the brine in the tank after t seconds is $V(t) = 100 + 2t$. Let us do an analysis similar to that in the book at page 51. Denote the amount of salt in the tank at time t by $y(t)$. We balance the change in salt $y(t+h) - y(t)$ during a small interval of time h in the following way: the difference comes from the amount of salt that is getting in the tank minus the amount that is getting out. The amount that is getting in is $5h$ lb/s, if we measure h in seconds. Then the amount that is getting out assuming perfect mixture (instantaneous) is approximately $\frac{y(t)}{V(t)}3h$ lb of salt. So, the balance is $\frac{y(t+h)-y(t)}{h} \approx 5 - \frac{3y(t)}{100+2t}$, $t \in [0, 150]$. Letting h go to zero, we obtain the initial value problem

$$\begin{cases} \frac{dy}{dt} = 5 - \frac{3y(t)}{100+2t} \\ y(0) = 50. \end{cases} \tag{1.28}$$

This is a linear equation with initial condition that we solve using the same method as above. We have $R(t) = \int_0^t \frac{3}{100+2s}ds = \frac{3}{2}\ln(\frac{50+t}{50})$. This means that we need to multiply by $(\frac{50+t}{50})^{\frac{3}{2}}$ both sides of $\frac{dy}{dt}+\frac{3y(t)}{100+2t}=5$. We get

$$\left(\frac{50+t}{50}\right)^{\frac{3}{2}}\frac{dy}{dt}+\frac{3}{100}\left(\frac{50+t}{50}\right)^{\frac{1}{2}}y(t)=5\left(\frac{50+t}{50}\right)^{\frac{3}{2}}$$

Integrating this last equation with respect to t from 0 to s, we obtain:

$$\left(\frac{50+s}{50}\right)^{\frac{3}{2}}y(s)-50=\frac{2}{50\sqrt{50}}((50+s)^{\frac{5}{2}}-50^2\sqrt{50}).$$

Substituting $s=150$ in this last equality, we obtain $y(150)=\frac{800000\sqrt{2}-12500\sqrt{2}}{2000\sqrt{2}}\approx$ 393.75 lb. ■

Homework:

Section 1.4 pages 41–44: 1-28, 32, 43, 48, 61, 62 and 64;

Section 1.5 pages 54-56: 11-15, 26-33.

The fundamental theorem of calculus therefore implies that $dV/dy = A(y)$ and hence that

$$\frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt} = A(y)\frac{dy}{dt}. \tag{29}$$

From Eqs. (28) and (29) we finally obtain

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy} = -k\sqrt{y}, \tag{30}$$

an alternative form of Torricelli's law.

Example 6 A hemispherical bowl has top radius 4 ft and at time $t = 0$ is full of water. At that moment a circular hole with diameter 1 in. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?

Solution From the right triangle in Fig. 1.4.9, we see that

$$A(y) = \pi r^2 = \pi\left[16 - (4-y)^2\right] = \pi(8y - y^2).$$

With $g = 32$ ft/s^2, Eq. (30) becomes

$$\pi(8y - y^2)\frac{dy}{dt} = -\pi\left(\tfrac{1}{24}\right)^2\sqrt{2 \cdot 32y};$$

$$\int (8y^{1/2} - y^{3/2})\,dy = -\int \tfrac{1}{72}\,dt;$$

$$\tfrac{16}{3}y^{3/2} - \tfrac{2}{5}y^{5/2} = -\tfrac{1}{72}t + C.$$

Now $y(0) = 4$, so

$$C = \tfrac{16}{3} \cdot 4^{3/2} - \tfrac{2}{5} \cdot 4^{5/2} = \tfrac{448}{15}.$$

The tank is empty when $y = 0$, thus when

$$t = 72 \cdot \tfrac{448}{15} \approx 2150 \text{ (s)};$$

that is, about 35 min 50 s. So it takes slightly less than 36 min for the tank to drain. ■

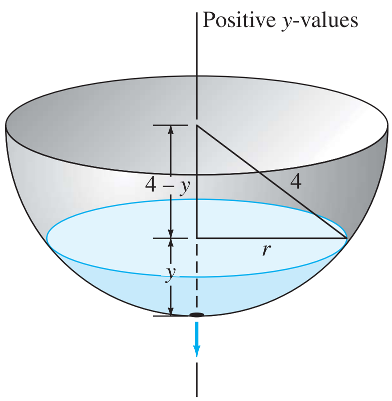


FIGURE 1.4.9. Draining a hemispherical tank.

1.4 Problems

Find general solutions (implicit if necessary, explicit if convenient) of the differential equations in Problems 1 through 18. Primes denote derivatives with respect to x.

1. $\dfrac{dy}{dx} + 2xy = 0$

2. $\dfrac{dy}{dx} + 2xy^2 = 0$

3. $\dfrac{dy}{dx} = y \sin x$

4. $(1+x)\dfrac{dy}{dx} = 4y$

5. $2\sqrt{x}\dfrac{dy}{dx} = \sqrt{1-y^2}$

6. $\dfrac{dy}{dx} = 3\sqrt{xy}$

7. $\dfrac{dy}{dx} = (64xy)^{1/3}$

8. $\dfrac{dy}{dx} = 2x \sec y$

9. $(1-x^2)\dfrac{dy}{dx} = 2y$

10. $(1+x)^2\dfrac{dy}{dx} = (1+y)^2$

11. $y' = xy^3$

12. $yy' = x(y^2+1)$

13. $y^3\dfrac{dy}{dx} = (y^4+1)\cos x$

14. $\dfrac{dy}{dx} = \dfrac{1+\sqrt{x}}{1+\sqrt{y}}$

15. $\dfrac{dy}{dx} = \dfrac{(x-1)y^5}{x^2(2y^3-y)}$

16. $(x^2+1)(\tan y)y' = x$

17. $y' = 1 + x + y + xy$ (*Suggestion:* Factor the right-hand side.)

18. $x^2y' = 1 - x^2 + y^2 - x^2y^2$

Find explicit particular solutions of the initial value problems in Problems 19 through 28.

19. $\dfrac{dy}{dx} = ye^x, \quad y(0) = 2e$

20. $\dfrac{dy}{dx} = 3x^2(y^2+1), \quad y(0) = 1$

21. $2y\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2-16}}, \quad y(5) = 2$

22. $\dfrac{dy}{dx} = 4x^3y - y, \quad y(1) = -3$

23. $\dfrac{dy}{dx} + 1 = 2y, \quad y(1) = 1$

24. $\dfrac{dy}{dx} = y \cot x, \quad y\left(\tfrac{1}{2}\pi\right) = \tfrac{1}{2}\pi$

25. $x\dfrac{dy}{dx} - y = 2x^2y, \quad y(1) = 1$

26. $\dfrac{dy}{dx} = 2xy^2 + 3x^2y^2, \quad y(1) = -1$

27. $\dfrac{dy}{dx} = 6e^{2x-y}, \quad y(0) = 0$

28. $2\sqrt{x}\dfrac{dy}{dx} = \cos^2 y, \quad y(4) = \pi/4$

29. **(a)** Find a general solution of the differential equation $dy/dx = y^2$. **(b)** Find a singular solution that is not included in the general solution. **(c)** Inspect a sketch of typical solution curves to determine the points (a, b) for which the initial value problem $y' = y^2$, $y(a) = b$ has a unique solution.

30. Solve the differential equation $(dy/dx)^2 = 4y$ to verify the general solution curves and singular solution curve that are illustrated in Fig. 1.4.5. Then determine the points (a, b) in the plane for which the initial value problem $(y')^2 = 4y$, $y(a) = b$ has **(a)** no solution, **(b)** infinitely many solutions that are defined for all x, **(c)** on some neighborhood of the point $x = a$, only finitely many solutions.

31. Discuss the difference between the differential equations $(dy/dx)^2 = 4y$ and $dy/dx = 2\sqrt{y}$. Do they have the same solution curves? Why or why not? Determine the points (a, b) in the plane for which the initial value problem $y' = 2\sqrt{y}$, $y(a) = b$ has **(a)** no solution, **(b)** a unique solution, **(c)** infinitely many solutions.

32. Find a general solution and any singular solutions of the differential equation $dy/dx = y\sqrt{y^2 - 1}$. Determine the points (a, b) in the plane for which the initial value problem $y' = y\sqrt{y^2 - 1}$, $y(a) = b$ has **(a)** no solution, **(b)** a unique solution, **(c)** infinitely many solutions.

33. (Population growth) A certain city had a population of 25,000 in 1960 and a population of 30,000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. What population can its city planners expect in the year 2000?

34. (Population growth) In a certain culture of bacteria, the number of bacteria increased sixfold in 10 h. How long did it take for the population to double?

35. (Radiocarbon dating) Carbon extracted from an ancient skull contained only one-sixth as much ^{14}C as carbon extracted from present-day bone. How old is the skull?

36. (Radiocarbon dating) Carbon taken from a purported relic of the time of Christ contained 4.6×10^{10} atoms of ^{14}C per gram. Carbon extracted from a present-day specimen of the same substance contained 5.0×10^{10} atoms of ^{14}C per gram. Compute the approximate age of the relic. What is your opinion as to its authenticity?

37. (Continuously compounded interest) Upon the birth of their first child, a couple deposited \$5000 in an account that pays 8% interest compounded continuously. The interest payments are allowed to accumulate. How much will the account contain on the child's eighteenth birthday?

38. (Continuously compounded interest) Suppose that you discover in your attic an overdue library book on which your grandfather owed a fine of 30 cents 100 years ago. If an overdue fine grows exponentially at a 5% annual rate compounded continuously, how much would you have to pay if you returned the book today?

39. (Drug elimination) Suppose that sodium pentobarbital is used to anesthetize a dog. The dog is anesthetized when its bloodstream contains at least 45 milligrams (mg) of sodium pentobarbitol per kilogram of the dog's body weight. Suppose also that sodium pentobarbitol is eliminated exponentially from the dog's bloodstream, with a half-life of 5 h. What single dose should be administered in order to anesthetize a 50-kg dog for 1 h?

40. The half-life of radioactive cobalt is 5.27 years. Suppose that a nuclear accident has left the level of cobalt radiation in a certain region at 100 times the level acceptable for human habitation. How long will it be until the region is again habitable? (Ignore the probable presence of other radioactive isotopes.)

41. Suppose that a mineral body formed in an ancient cataclysm—perhaps the formation of the earth itself—originally contained the uranium isotope ^{238}U (which has a half-life of 4.51×10^9 years) but no lead, the end product of the radioactive decay of ^{238}U. If today the ratio of ^{238}U atoms to lead atoms in the mineral body is 0.9, when did the cataclysm occur?

42. A certain moon rock was found to contain equal numbers of potassium and argon atoms. Assume that all the argon is the result of radioactive decay of potassium (its half-life is about 1.28×10^9 years) and that one of every nine potassium atom disintegrations yields an argon atom. What is the age of the rock, measured from the time it contained only potassium?

43. A pitcher of buttermilk initially at 25°C is to be cooled by setting it on the front porch, where the temperature is 0°C. Suppose that the temperature of the buttermilk has dropped to 15°C after 20 min. When will it be at 5°C?

44. When sugar is dissolved in water, the amount A that remains undissolved after t minutes satisfies the differential equation $dA/dt = -kA$ $(k > 0)$. If 25% of the sugar dissolves after 1 min, how long does it take for half of the sugar to dissolve?

45. The intensity I of light at a depth of x meters below the surface of a lake satisfies the differential equation $dI/dx = (-1.4)I$. **(a)** At what depth is the intensity half the intensity I_0 at the surface (where $x = 0$)? **(b)** What is the intensity at a depth of 10 m (as a fraction of I_0)? **(c)** At what depth will the intensity be 1% of that at the surface?

46. The barometric pressure p (in inches of mercury) at an altitude x miles above sea level satisfies the initial value problem $dp/dx = (-0.2)p$, $p(0) = 29.92$. **(a)** Calculate the barometric pressure at 10,000 ft and again at 30,000 ft. **(b)** Without prior conditioning, few people can survive when the pressure drops to less than 15 in. of mercury. How high is that?

47. A certain piece of dubious information about phenylethylamine in the drinking water began to spread one day in a city with a population of 100,000. Within a week, 10,000 people had heard this rumor. Assume that the rate of increase of the number who have heard the rumor is proportional to the number who have not yet heard it. How long will it be until half the population of the city has heard the rumor?

48. According to one cosmological theory, there were equal amounts of the two uranium isotopes ^{235}U and ^{238}U at the creation of the universe in the "big bang." At present there are 137.7 atoms of ^{238}U for each atom of ^{235}U. Using the half-lives 4.51×10^9 years for ^{238}U and 7.10×10^8 years for ^{235}U, calculate the age of the universe.

49. A cake is removed from an oven at 210°F and left to cool at room temperature, which is 70°F. After 30 min the temperature of the cake is 140°F. When will it be 100°F?

50. The amount $A(t)$ of atmospheric pollutants in a certain mountain valley grows naturally and is tripling every 7.5 years.

(a) If the initial amount is 10 pu (pollutant units), write a formula for $A(t)$ giving the amount (in pu) present after t years.

(b) What will be the amount (in pu) of pollutants present in the valley atmosphere after 5 years?

(c) If it will be dangerous to stay in the valley when the amount of pollutants reaches 100 pu, how long will this take?

51. An accident at a nuclear power plant has left the surrounding area polluted with radioactive material that decays naturally. The initial amount of radioactive material present is 15 su (safe units), and 5 months later it is still 10 su.

(a) Write a formula giving the amount $A(t)$ of radioactive material (in su) remaining after t months.

(b) What amount of radioactive material will remain after 8 months?

(c) How long—total number of months or fraction thereof—will it be until $A = 1$ su, so it is safe for people to return to the area?

52. There are now about 3300 different human "language families" in the whole world. Assume that all these are derived from a single original language and that a language family develops into 1.5 language families every 6 thousand years. About how long ago was the single original human language spoken?

53. Thousands of years ago ancestors of the Native Americans crossed the Bering Strait from Asia and entered the western hemisphere. Since then, they have fanned out across North and South America. The single language that the original Native Americans spoke has since split into many Indian "language families." Assume (as in Problem 52) that the number of these language families has been multiplied by 1.5 every 6000 years. There are now 150 Native American language families in the western hemisphere. About when did the ancestors of today's Native Americans arrive?

54. A tank is shaped like a vertical cylinder; it initially contains water to a depth of 9 ft, and a bottom plug is removed at time $t = 0$ (hours). After 1 h the depth of the water has dropped to 4 ft. How long does it take for all the water to drain from the tank?

55. Suppose that the tank of Problem 54 has a radius of 3 ft and that its bottom hole is circular with radius 1 in. How long will it take the water (initially 9 ft deep) to drain completely?

56. At time $t = 0$ the bottom plug (at the vertex) of a full conical water tank 16 ft high is removed. After 1 h the water in the tank is 9 ft deep. When will the tank be empty?

57. Suppose that a cylindrical tank initially containing V_0 gallons of water drains (through a bottom hole) in T minutes. Use Torricelli's law to show that the volume of water in the tank after $t \leqq T$ minutes is $V = V_0\,[1 - (t/T)]^2$.

58. A water tank has the shape obtained by revolving the curve $y = x^{4/3}$ around the y-axis. A plug at the bottom is removed at 12 noon, when the depth of water in the tank is 12 ft. At 1 P.M. the depth of the water is 6 ft. When will the tank be empty?

59. A water tank has the shape obtained by revolving the parabola $x^2 = by$ around the y-axis. The water depth is 4 ft at 12 noon, when a circular plug in the bottom of the tank is removed. At 1 P.M. the depth of the water is 1 ft. **(a)** Find the depth $y(t)$ of water remaining after t hours. **(b)** When will the tank be empty? **(c)** If the initial radius of the top surface of the water is 2 ft, what is the radius of the circular hole in the bottom?

60. A cylindrical tank with length 5 ft and radius 3 ft is situated with its axis horizontal. If a circular bottom hole with a radius of 1 in. is opened and the tank is initially half full of water, how long will it take for the liquid to drain completely?

61. A spherical tank of radius 4 ft is full of water when a circular bottom hole with radius 1 in. is opened. How long will be required for all the water to drain from the tank?

62. Suppose that an initially full hemispherical water tank of radius 1 m has its flat side as its bottom. It has a bottom hole of radius 1 cm. If this bottom hole is opened at 1 P.M., when will the tank be empty?

63. Consider the initially full hemispherical water tank of Example 8, except that the radius r of its circular bottom hole is now unknown. At 1 P.M. the bottom hole is opened and at 1:30 P.M. the depth of water in the tank is 2 ft. **(a)** Use Torricelli's law in the form $dV/dt = -(0.6)\pi r^2\sqrt{2gy}$ (taking constriction into account) to determine when the tank will be empty. **(b)** What is the radius of the bottom hole?

64. (The *clepsydra*, or water clock) A 12 h water clock is to be designed with the dimensions shown in Fig. 1.4.10, shaped like the surface obtained by revolving the curve

$y = f(x)$ around the y-axis. What should this curve be, and what should the radius of the circular bottom hole be, in order that the water level will fall at the *constant* rate of 4 inches per hour (in./h)?

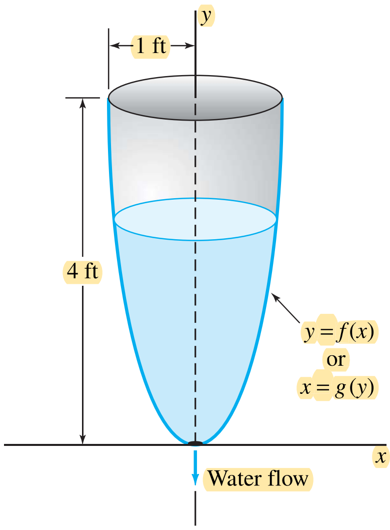


FIGURE 1.4.10. The clepsydra.

65. Just before midday the body of an apparent homicide victim is found in a room that is kept at a constant temperature of 70°F. At 12 noon the temperature of the body is 80°F and at 1 P.M. it is 75°F. Assume that the temperature of the body at the time of death was 98.6°F and that it has cooled in accord with Newton's law. What was the time of death?

66. Early one morning it began to snow at a constant rate. At 7 A.M. a snowplow set off to clear a road. By 8 A.M. it had traveled 2 miles, but it took two more hours (until 10 A.M.) for the snowplow to go an additional 2 miles. **(a)** Let $t = 0$ when it began to snow, and let x denote the distance traveled by the snowplow at time t. Assuming that the snowplow clears snow from the road at a constant rate (in cubic feet per hour, say), show that

$$k\frac{dx}{dt} = \frac{1}{t}$$

where k is a constant. **(b)** What time did it start snowing? (*Answer:* 6 A.M.)

67. A snowplow sets off at 7 A.M. as in Problem 66. Suppose now that by 8 A.M. it had traveled 4 miles and that by 9 A.M. it had moved an additional 3 miles. What time did it start snowing? This is a more difficult snowplow problem because now a transcendental equation must be solved numerically to find the value of k. (*Answer:* 4:27 A.M.)

68. Figure 1.4.11 shows a bead sliding down a frictionless wire from point P to point Q. The *brachistochrone problem* asks what shape the wire should be in order to minimize the bead's time of descent from P to Q. In June of 1696, John Bernoulli proposed this problem as a public challenge, with a 6-month deadline (later extended to Easter 1697 at George Leibniz's request). Isaac Newton, then retired from academic life and serving as Warden of the Mint in London, received Bernoulli's challenge on January 29, 1697. The very next day he communicated his own solution—the curve of minimal descent time is an arc of an inverted cycloid—to the Royal Society of London. For a modern derivation of this result, suppose the bead starts from rest at the origin P and let $y = y(x)$ be the equation of the desired curve in a coordinate system with the y-axis pointing downward. Then a mechanical analogue of Snell's law in optics implies that

$$\frac{\sin\alpha}{v} = \text{constant}, \qquad \text{(i)}$$

where α denotes the angle of deflection (from the vertical) of the tangent line to the curve—so $\cot\alpha = y'(x)$ (why?)—and $v = \sqrt{2gy}$ is the bead's velocity when it has descended a distance y vertically (from $\text{KE} = \frac{1}{2}mv^2 = mgy = -\text{PE}$).

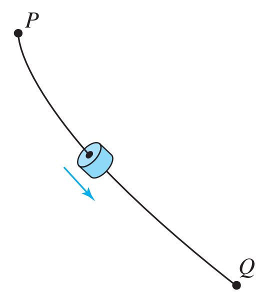


FIGURE 1.4.11. A bead sliding down a wire—the brachistochrone problem.

(a) First derive from Eq. (i) the differential equation

$$\frac{dy}{dx} = \sqrt{\frac{2a-y}{y}}, \qquad \text{(ii)}$$

where a is an appropriate positive constant.

(b) Substitute $y = 2a\sin^2 t$, $dy = 4a\sin t\cos t\,dt$ in (ii) to derive the solution

$$x = a(2t - \sin 2t), \quad y = a(1 - \cos 2t) \qquad \text{(iii)}$$

for which $t = y = 0$ when $x = 0$. Finally, the substitution of $\theta = 2t$ in (iii) yields the standard parametric equations $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ of the cycloid that is generated by a point on the rim of a circular wheel of radius a as it rolls along the x-axis. [See Example 5 in Section 9.4 of Edwards and Penney, *Calculus: Early Transcendentals*, 7th edition (Upper Saddle River, NJ: Prentice Hall, 2008).]

69. Suppose a uniform flexible cable is suspended between two points $(\pm L, H)$ at equal heights located symmetrically on either side of the x-axis (Fig. 1.4.12). Principles of physics can be used to show that the shape $y = y(x)$ of the hanging cable satisfies the differential equation

$$a\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

where the constant $a = T/\rho$ is the ratio of the cable's tension T at its lowest point $x = 0$ (where $y'(0) = 0$) and

Substitution of $x(0) = 90$ gives $C = -(90)^4$, so the amount of salt in the tank at time t is

$$x(t) = 2(90+t) - \frac{90^4}{(90+t)^3}.$$

The tank is full after 30 min, and when $t = 30$, we have

$$x(30) = 2(90+30) - \frac{90^4}{120^3} \approx 202 \text{ (lb)}$$

of salt in the tank. ■

1.5 Problems

Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.

1. $y' + y = 2$, $y(0) = 0$

2. $y' - 2y = 3e^{2x}$, $y(0) = 0$

3. $y' + 3y = 2xe^{-3x}$

4. $y' - 2xy = e^{x^2}$

5. $xy' + 2y = 3x$, $y(1) = 5$

6. $xy' + 5y = 7x^2$, $y(2) = 5$

7. $2xy' + y = 10\sqrt{x}$

8. $3xy' + y = 12x$

9. $xy' - y = x$, $y(1) = 7$

10. $2xy' - 3y = 9x^3$

11. $xy' + y = 3xy$, $y(1) = 0$

12. $xy' + 3y = 2x^5$, $y(2) = 1$

13. $y' + y = e^x$, $y(0) = 1$

14. $xy' - 3y = x^3$, $y(1) = 10$

15. $y' + 2xy = x$, $y(0) = -2$

16. $y' = (1 - y)\cos x$, $y(\pi) = 2$

17. $(1 + x)y' + y = \cos x$, $y(0) = 1$

18. $xy' = 2y + x^3 \cos x$

19. $y' + y \cot x = \cos x$

20. $y' = 1 + x + y + xy$, $y(0) = 0$

21. $xy' = 3y + x^4 \cos x$, $y(2\pi) = 0$

22. $y' = 2xy + 3x^2 \exp(x^2)$, $y(0) = 5$

23. $xy' + (2x - 3)y = 4x^4$

24. $(x^2 + 4)y' + 3xy = x$, $y(0) = 1$

25. $(x^2 + 1)\dfrac{dy}{dx} + 3x^3 y = 6x \exp\left(-\frac{3}{2}x^2\right)$, $y(0) = 1$

Solve the differential equations in Problems 26 through 28 by regarding y as the independent variable rather than x.

26. $(1 - 4xy^2)\dfrac{dy}{dx} = y^3$

27. $(x + ye^y)\dfrac{dy}{dx} = 1$

28. $(1 + 2xy)\dfrac{dy}{dx} = 1 + y^2$

29. Express the general solution of $dy/dx = 1 + 2xy$ in terms of the **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

30. Express the solution of the initial value problem

$$2x\frac{dy}{dx} = y + 2x\cos x, \quad y(1) = 0$$

as an integral as in Example 3 of this section.

Problems 31 and 32 illustrate—for the special case of first-order linear equations—techniques that will be important when we study higher-order linear equations in Chapter 3.

31. **(a)** Show that

$$y_c(x) = Ce^{-\int P(x)\,dx}$$

is a general solution of $dy/dx + P(x)y = 0$. **(b)** Show that

$$y_p(x) = e^{-\int P(x)\,dx}\left[\int \left(Q(x)e^{\int P(x)\,dx}\right) dx\right]$$

is a particular solution of $dy/dx + P(x)y = Q(x)$. **(c)** Suppose that $y_c(x)$ is any general solution of $dy/dx + P(x)y = 0$ and that $y_p(x)$ is any particular solution of $dy/dx + P(x)y = Q(x)$. Show that $y(x) = y_c(x) + y_p(x)$ is a general solution of $dy/dx + P(x)y = Q(x)$.

32. **(a)** Find constants A and B such that $y_p(x) = A\sin x + B\cos x$ is a solution of $dy/dx + y = 2\sin x$. **(b)** Use the result of part (a) and the method of Problem 31 to find the general solution of $dy/dx + y = 2\sin x$. **(c)** Solve the initial value problem $dy/dx + y = 2\sin x$, $y(0) = 1$.

33. A tank contains 1000 liters (L) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture—kept uniform by stirring— is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

34. Consider a reservoir with a volume of 8 billion cubic feet (ft^3) and an initial pollutant concentration of 0.25%. There is a daily inflow of 500 million ft^3 of water with a pollutant concentration of 0.05% and an equal daily outflow of the well-mixed water in the reservoir. How long will it take to reduce the pollutant concentration in the reservoir to 0.10%?

35. Rework Example 4 for the case of Lake Ontario, which empties into the St. Lawrence River and receives inflow from Lake Erie (via the Niagara River). The only differences are that this lake has a volume of 1640 km^3 and an inflow-outflow rate of 410 km^3/year.

36. A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus the tank is empty after exactly 1 h. **(a)** Find the amount of salt in the tank after t minutes. **(b)** What is the maximum amount of salt ever in the tank?

37. A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

38. Consider the *cascade* of two tanks shown in Fig. 1.5.5, with $V_1 = 100$ (gal) and $V_2 = 200$ (gal) the volumes of brine in the two tanks. Each tank also initially contains 50 lb of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank 1. **(a)** Find the amount $x(t)$ of salt in tank 1 at time t. **(b)** Suppose that $y(t)$ is the amount of salt in tank 2 at time t. Show first that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200},$$

and then solve for $y(t)$, using the function $x(t)$ found in part (a). **(c)** Finally, find the maximum amount of salt ever in tank 2.

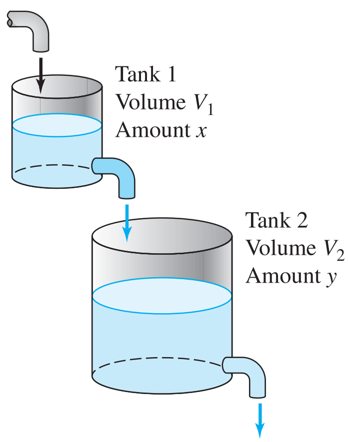


FIGURE 1.5.5. A cascade of two tanks.

39. Suppose that in the cascade shown in Fig. 1.5.5, tank 1 initially contains 100 gal of pure ethanol and tank 2 initially contains 100 gal of pure water. Pure water flows into tank 1 at 10 gal/min, and the other two flow rates are also 10 gal/min. **(a)** Find the amounts $x(t)$ and $y(t)$ of ethanol in the two tanks at time $t \geqq 0$. **(b)** Find the maximum amount of ethanol ever in tank 2.

40. A multiple cascade is shown in Fig. 1.5.6. At time $t = 0$, tank 0 contains 1 gal of ethanol and 1 gal of water; all the remaining tanks contain 2 gal of pure water each. Pure water is pumped into tank 0 at 1 gal/min, and the varying mixture in each tank is pumped into the one below it at the same rate. Assume, as usual, that the mixtures are kept perfectly uniform by stirring. Let $x_n(t)$ denote the amount of ethanol in tank n at time t.

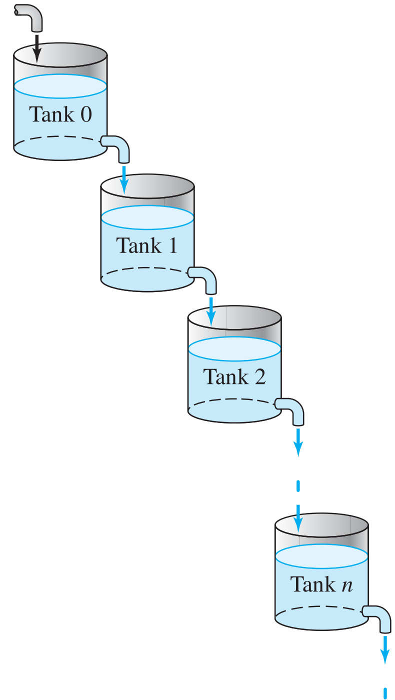


FIGURE 1.5.6. A multiple cascade.

(a) Show that $x_0(t) = e^{-t/2}$. **(b)** Show by induction on n that

$$x_n(t) = \frac{t^n e^{-t/2}}{n!\,2^n} \quad \text{for } n \geq 0.$$

(c) Show that the maximum value of $x_n(t)$ for $n > 0$ is $M_n = x_n(2n) = n^n e^{-n}/n!$. **(d)** Conclude from **Stirling's approximation** $n! \approx n^n e^{-n}\sqrt{2\pi n}$ that $M_n \approx (2\pi n)^{-1/2}$.

41. A 30-year-old woman accepts an engineering position with a starting salary of \$30,000 per year. Her salary $S(t)$ increases exponentially, with $S(t) = 30e^{t/20}$ thousand dollars after t years. Meanwhile, 12% of her salary is deposited continuously in a retirement account, which accumulates interest at a continuous annual rate of 6%. **(a)** Estimate ΔA in terms of Δt to derive the differential equation satisfied by the amount $A(t)$ in her retirement account after t years. **(b)** Compute $A(40)$, the amount available for her retirement at age 70.

42. Suppose that a falling hailstone with density $\delta = 1$ starts from rest with negligible radius $r = 0$. Thereafter its radius is $r = kt$ (k is a constant) as it grows by accretion during its fall. Use Newton's second law—according to which the net force F acting on a possibly variable mass m equals the time rate of change dp/dt of its momentum $p = mv$—to set up and solve the initial value problem

$$\frac{d}{dt}(mv) = mg, \quad v(0) = 0,$$

where m is the variable mass of the hailstone, $v = dy/dt$ is its velocity, and the positive y-axis points downward. Then show that $dv/dt = g/4$. Thus the hailstone falls as though it were under *one-fourth* the influence of gravity.

43. Figure 1.5.7 shows a slope field and typical solution curves for the equation $y' = x - y$. **(a)** Show that every solution curve approaches the straight line $y = x - 1$

1.4 Lecture IV

Quotation: *"The highest form of pure thought is in mathematics", Plato*

Other type of equations which can be solved with exact methods, notions, real world applications: *homogeneous (in terms of variables) differential equations, Bernoulli differential equations, exact differential equations, characterization of exactness, reducible to second-order DE (dependent variable missing, independent variable missing).*

1.4.1 Substitution methods

We begin with an example by solving the Problem 55, page 72.

Problem 1.4.1. *Show that the substitution $v = ax+by+c$ transforms the differential equation $\frac{dy}{dx} = F(ax+by+c)$ into a separable equation.*

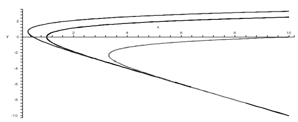
Solution: Let us assume that $b \neq 0$. Differentiating the substitution $v = ax+by+c$ with respect to x, we get $\frac{dv}{dx} = a+b\frac{dy}{dx}$ and then the equation becomes $\frac{dv}{dx} = a+bF(v)$ which is a separable equation indeed. If $b = 0$ then the equation $\frac{dy}{dx} = F(ax+c)$ is already separable. ■

As an application lets work problem 18 on page 71: *find a general solution of* $(x+y)y' = 1$. We make the substitution $v = x+y$. Then $\frac{dv}{dx} = 1 + \frac{dy}{dx}$ which turns the original equation into $v(v'-1) = 1$ or $vv' = v+1$. Observe that one singular solution of this equation is $v(x) = -1$ for all $x \in \mathbb{R}$ which corresponds to $y(x) = -x-1$ for all $x \in \mathbb{R}$.

If $v(x) \neq -1$ for x in some interval we can write the equation as $\frac{vv'}{v+1} = 1$. Equivalently, in order to integrate let us write this equation as

$$v' - \frac{v'}{v+1} = 1.$$

Integrating with respect to x we obtain $v - \ln|v+1| = x + C$. Getting back to the original variable this can be written as $x+y-\ln|x+y+1| = x+C$. Notice that this could be simplified to $y = \ln|x+y+1| + C$ which gives y only implicitly. If we want to get rid of the logarithmic function and also of the absolute value function as well, we can exponentiate the last equality to turn it into $e^y = k(x+y+1)$ where k is a real constant which is not zero. In order to include the singular solution we can move the constant to the other side and allow it to be zero: $x+y+1 = k_1e^y$, $k_1 \in \mathbb{R}$. Some solutions curves can be drawn with Maple and we include some here ($k = 1, 2$ and $1/10$).



$k(x+y+1)=e^y,\ k=1,2,\frac{1}{10}$

1.5 Homogeneous DE

The homogeneous property referes here to the function f when writing the DE as $y'=f(x,y)$. The prototype is actually

$$y'=f(y/x).$$

The recommended substitution here is $v=y/x$. This implies $y(x)=xv(x)$ so, differentiating with respect to x we obtain $y'=v+xv'$. Then the original equation becomes $v'=(f(v)-v)/x$ which is a separable DE.

In order to check that a differential equation is homogeneous we could substitute $y=vx$ in the expression of $f(x,y)$ and see if the resulting function can be written just in terms of v. As an example let's take Exercise 14, page 71.

Problem 1.5.1. *Find all solutions of the equation $yy'+x=\sqrt{x^2+y^2}$.*

Solution: One can check that this DE is homogeneous. Let $y=xv$, where v is a function of x. Then $y'=v+xv'$ and so our equation becomes $xv(v+xv')+x=\sqrt{x^2+x^2v^2}$. Assume first that we are working on an interval $I\subset(0,\infty)$. Then the DE simplifies to $xvv'=\sqrt{1+v^2}-1-v^2$. Let us observe that $v(x)=0$ for all $x\in I$ is a solution of this equation. We will see that this is not a singular solution. So, if we assume v is not zero on I, say $v>0$, the DE is equivalent to $\frac{vv'}{1+v^2-\sqrt{1+v^2}}=-\frac{1}{x}$ or $\frac{vv'}{\sqrt{1+v^2}(\sqrt{1+v^2}-1)}=-\frac{1}{x}$.

Using the conjugate we can modify the left hand side to $\frac{v(\sqrt{1+v^2}+1)v'}{v^2\sqrt{1+v^2}}=-\frac{1}{x}$ or

$$\frac{v'}{v}+\frac{v'}{v\sqrt{1+v^2}}=-\frac{1}{x}. \tag{1.29}$$

If we integrate both sides with respect to x we obtain $\ln(v)+\int\frac{dv}{v\sqrt{1+v^2}}=-\ln x+C$. In the indefinite integral we've got, let us do the change of variables: $v=\frac{1}{u}$. This integral becomes $\int\frac{dv}{v\sqrt{1+v^2}}=-\int\frac{du}{\sqrt{1+u^2}}=-\ln|u+\sqrt{1+u^2}|+C'=\ln|v|-\ln(\sqrt{1+v^2}+1)+C'$. Hence the DE (1.29) leads to

$$\frac{xv^2}{\sqrt{1+v^2}+1}=k \tag{1.30}$$

where the constant k can take any non-negative value. Using the conjugate again (2) changes into $x(\sqrt{1+v^2}-1)=k$ or $\sqrt{x^2+y^2}-x=k$. If we solve this for y we obtain

$$y(x)=\sqrt{(2x+k)k} \tag{1.31}$$

Let us observe that this function is actually defined on $(-k/2,\infty)$ if $k\geq 0$. One can go back and check that $v<0$ leads to the choice of

$$y(x)=-\sqrt{(2x+k)k},\ x\in(-k/2,\infty). \tag{1.32}$$

The two general solutions (1.31) and (1.32) are all the solutions of the original equation (the singular solution $y(x)=0$ is included in (1.31) for $k=0$.

$\blacksquare$

1.6 Bernoulli DE

The general form of these equations is very close to that of linear DE:

$$y'+P(x)y=Q(x)y^n. \tag{1.33}$$

So, we may assume that n is different of 0 or 1 since these cases lead to DE that we have already studied. The recommended substitution is $v=y^{1-n}$. This implies $y=v^{\frac{1}{1-n}}$ and $y'=\frac{1}{1-n}v^{n/(1-n)}v'$. This changes the original equation to

$$\frac{1}{1-n}v^{\frac{n}{1-n}}v'+P(x)v^{\frac{1}{1-n}}=Q(x)v^{\frac{n}{1-n}},$$

which after dividing by $v^{\frac{n}{1-n}}$ (assuming is not zero)

$$\frac{1}{1-n}v'+P(x)v=Q(x),$$

which is a linear DE.

Let us see how this method works with the exercise 26, page 71.

Problem 1.6.1. *Find all solutions of the equation* $3y^2y' + y^3 = e^{-x}$.

Solution: If we put the given equation in the form (1.33) we get $y' + \frac{1}{3}y = \frac{e^{-x}}{3}y^{-2}$. This says that $n = -2$ and so we substitute $v = y^3$. This could have been observed from the start. The equation becomes $v' + v = e^{-x}$ or $(e^x v(x))' = 1$. Hence $e^x v(x) = x + C$ and then $v(x) = (C + x)e^{-x}$. This gives the general solution $y(x) = (C + x)^{1/3}e^{-x/3}$ for all $x \in \mathbb{R}$.

1.6.1 Exact equations

We say the equation

$$y' = f(x, y) \tag{1.34}$$

is an *exact equation* if $f(x, y) = -\dfrac{M(x, y)}{N(x, y)}$ and for some function $F(x, y)$ we have $M(x, y) = \dfrac{\partial F}{\partial x}(x, y)$ and $N(x, y) = \dfrac{\partial F}{\partial y}(x, y)$. Let us observe that if an equation is exact then $F(x, y(x)) = C$ is a general solution giving y implicitly: indeed, differentiating this with respect to x we get $\frac{\partial F}{\partial x}(x, y) + \frac{\partial F}{\partial y}(x, y)y'(x) = 0$. This is nothing but the original equation. There is a condition on M and N that tells us if the equation is exact or not.

Theorem 1.6.2. *Let M and N as before, defined and continuously differentiable on a rectangle $\mathcal{R} = \{(x, y) : a < x < b, c < y < d\}$. Then the equation (1.34) is exact if and only if $\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$ for each $(x, y) \in \mathcal{R}$.*

Proof: First suppose that the equation (1.34) is exact. Then there exists F differentiable such that $M(x, y) = \dfrac{\partial F}{\partial x}(x, y)$ and $N(x, y) = \dfrac{\partial F}{\partial y}(x, y)$ with $f(x, y) = -\dfrac{M(x, y)}{N(x, y)}$. Then $\dfrac{\partial M}{\partial y}(x, y) = \dfrac{\partial^2 F}{\partial y \partial x}(x, y)$ and $\dfrac{\partial N}{\partial x}(x, y) = \dfrac{\partial^2 F}{\partial x \partial y}(x, y)$. For a function which twice continuously differentiable the mixed derivatives are equal: $\dfrac{\partial^2 F}{\partial y \partial x}(x, y) = \dfrac{\partial^2 F}{\partial x \partial y}(x, y)$. This is called Schwartz (or Clairaut)'s theorem.

For the other implication, we define $F(x,y)=\int_{y_0}^{y}N(x_0,s)ds+\int_{x_0}^{x}M(s,y)ds$. We need to check that $M(x,y)=\frac{\partial F}{\partial x}(x,y)$ and $N(x,y)=\frac{\partial F}{\partial y}(x,y)$. The first equality is a consequence of the Second Fundamental Theorem of Calculus since the first part in the definition of F is constant with respect to x and the second gives $\frac{\partial F}{\partial x}=\frac{d}{dx}[\int_{x_0}^{x}M(s,y)ds]=M(x,y)$. To check the second equality we differentiate with respect to y the definition of F: $\frac{\partial F}{\partial y}=\frac{d}{dy}[\int_{y_0}^{y}M(y_0,s)ds]+\int_{x_0}^{x}\frac{\partial M}{\partial y}(s,y)ds$. (We have used differentiation under the sign of integration which is true under our assumptions.) Using the hypothesis that $\frac{\partial M}{\partial y}(x,y)=\frac{\partial N}{\partial x}(x,y)$ we get $\frac{\partial F}{\partial y}=N(y_0,y)+\int_{x_0}^{x}\frac{\partial N}{\partial x}(s,y)ds=N(y_0,y)+N(x,y)-N(x_0,y)=N(x,y)$. ■

Example: Let us look at the problem 36, page 72. In this case $M(x,y)=(1+ye^{xy})$ and $N(x,y)=2y+xe^{xy}$. We need to check if the equation $M(x,y)+N(x,y)y'=0$ is exact. Using the Theorem 1.6.2 we see that we have to check that $\frac{\partial M}{\partial y}(x,y)=\frac{\partial N}{\partial x}(x,y)$. So, $\frac{\partial M}{\partial y}(x,y)=(1+xy)e^{xy}$ and $\frac{\partial N}{\partial x}(x,y)=(1+xy)e^{xy}$. In order to solve it we use the same method as in the proof of Theorem 3.1. First we integrate $M(x,y)$ with respect to x and obtain $F(x,y)=x+e^{xy}+h(y)$. Then we differentiate this with respect to y and obtain $N(x,y)=2y+xe^{xy}=xe^{xy}+h'(y)$. Hence $h(y)=2y$ which implies $h(y)=y^2+C$. Then the general solution of this given equation is $x+y^2+e^{xy}+C=0$.

1.6.2 Reducible second order DE

The DE we are dealing with in these cases is of the form

$$F(x,y,y',y'')=0. \tag{1.35}$$

In some situations just by making a substitution we can reduce this equation to a first order one. Case I. (*Dependent variable y missing*) In this case we substitute $v=y'$ Then the equation becomes a first-order equation.

Case II. (*Independent variable missing*) If the equation is written as $F(y,y',y'')=0$ then the substitution $v=y'$ will give $y''=\frac{dv}{dy}\frac{dy}{dx}=v\frac{dv}{dy}$ and the equation becomes $F(y,v,v\frac{dv}{dy})=0$ which is first-order DE.

Let us see an example like this. Problem 54, page 72 asks to solve the DE $yy''=3(y')^2$. If we substitute $v=y'$, $y''=v\frac{dv}{dy}$ we get $yv\frac{dv}{dy}=3v^2$. One particular solution of this equation is $v(x)=0$ for all x. Assuming that $v(x)\neq 0$ for x in some interval I we get $\frac{v'}{v} = \frac{3}{y}$. Integrating with respect to y we obtain $\ln|v| = \ln|y|^3 + C$. From here $v(y) = ky^3$ with k and arbitrary real constant. Since $y' = ky^3$ we can integrate again since this is a separable equation to obtain $-\frac{1}{2y^2} = kx + m$ where m is another constant. This gives the general solution $y(x) = \frac{C_1}{\sqrt{1+C_2x}}$ with $C_1, C_2 \in \mathbb{R}$.

Homework:

Section 1.6 pages 71–73: 1-23, 34-36, 43-55, 63 and 66;

Chapter 2

Analytical Methods, Second and n-order Linear Differential Equations

2.1 Lecture V

Quotation: *"I could never resist a definite integral." G.H. Hardy*

2.1.1 Stability, Euler's Method, Numerical Methods, Applications

Equilibrium solutions and stability for first-order autonomous DE, critical points, stable and instable critical points, bifurcation point, bifurcation diagram, vertical motion of a body with resistance proportional to velocity, Euler's approximation method, the error theorem in Euler's method

Another way of studying differential equations is to use qualitative methods in which one can say various things about a particular solution of the DE in question without necessarily solving for the solution in closed form or even in implicit form. Even for very simple differential equations which are autonomous first-order:

$$y' = f(y) \tag{2.1}$$

for some continuous function f, which leads to a separable DE, the integration $\int \frac{dy}{f(y)}$ may turn out to be very difficult. Not only that but these integrals may be impossible to be expressed in terms of elementary functions that we have reviewed earlier (polynomials, power functions, exponential and logarithmic ones, trigonometric functions and inverses of them).

One important concept for such an analysis in the case of DE of type (2.1) is the following notion:

Definition 2.1.1. *A number c for which $f(c) = 0$ is called a* **critical** *point of the DE (2.1).*

If c is a critical point for (2.1), we have a particular solution of (2.1): $y(x) = c$ for all $x \in I$. Such a solution is called an **equilibrium solution** of (2.1). For the following concept let us assume that f is also continuously differentiable on its domain of definition so that the existence and uniqueness theorem of Cauchy applies.

Definition 2.1.2. *A critical point c of (2.1) is said to be* **stable** *if for every $\epsilon > 0$ there exist a positive number δ such that if $|y_0 - c| \leq \delta$ then the solution of $y(x)$ of the initial value problem associated to (2.1)*

$$\begin{cases} y' = f(y) \\ y(0) = y_0 \end{cases} \tag{2.2}$$

satisfies $|y(x) - c| \leq \epsilon$ for all $x \geq 0$ and in the domain of the solution.

Notice that this is a very technical mathematical definition which is saying that if the initial point where a solution starts is close enough of the critical point c then the whole solution is going to stay close to the corresponding equilibrium solution at any other point in time (black whole behavior). If this definition is not satisfied we say that c is **unstable**.

Let us work the Problem 22, page 97 as we introduce all these related concepts and techniques.

Problem 2.1.3. *Consider the DE $y' = y + ky^3$ where k is a parameter. Determine the critical points and classify them as stable or unstable.*

Solution: The equation $y + ky^3 = 0$ has in general three solutions $y_1 = 0$, $y_{2,3} = \pm\sqrt{-1/k}$. There is only one solution if $k = 0$, only one real solution if $k > 0$ and three real ones if $k < 0$.

If $k = 0$ there is only one solution $y = 0$ and if y_0 is given the initial value problem (2.2) has unique solution $y(x) = y_0 e^x$, $x \in \mathbb{R}$, which has the property $\lim\limits_{x\to\infty} |y(x)| = \infty$, if $y_0 \neq 0$, which shows that $y = 0$ is unstable.

If $k > 0$ then $y'(x) \geq y(x)$ if $y(x) > 0$ at least. Then a similar method to that of solving linear equations shows that $y'(x) \geq y_0 e^x$ and this implies the $y = 0$ is unstable too. In fact one can integrate (2) and check that this is true. The general solution of

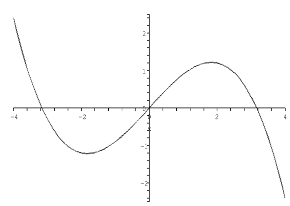
$$\begin{cases} y' = y + ky^3 \\ y(0) = y_0 \end{cases} \tag{2.3}$$

is given by $y(x) = \frac{y_0 e^x}{\sqrt{1 - ky_0^2(e^{2x}-1)}}$ (please check!). This solution is defined only for x satisfying $(e^{2x} - 1)ky_0^2 < 1$. So if $y_0 \neq 0$ then $x \in (-\infty, T)$ where $T = \frac{1}{2}\ln(1 + \frac{1}{ky_0^2})$. Let us observe that $\lim_{x\to T} |y(x)| = \infty$ so the point 0 is indeed unstable.

If $k < 0$ then the solution is well defined for all values of $x \in [0, \infty)$. We can write the expression of $y(x)$ in the form

$$y(x) = \frac{y_0}{\sqrt{e^{-2x}(1 + ky_0^2) - ky_0^2}}$$

which at the limit, as $x \to \infty$, is $\pm\sqrt{\frac{1}{-k}}$ depending upon y_0 is positive or negative. This shows that $y = 0$ is still unstable. On the other hand, one can check that, for instance, $|y(t) - \sqrt{\frac{1}{-k}}| \leq \epsilon$ if $|y_0 - \sqrt{\frac{1}{-k}}| \leq \delta$ where δ is chosen to be smaller than $\frac{1}{2a}$ and $\frac{a\epsilon}{14}$ $(a = \sqrt{-k})$. Similarly for the solution $-\sqrt{\frac{1}{-k}}$ which shows that both these critical points are stable. One could come to the same conclusion without solving the system (2.3). For $k < 0$ the graph of $f(y) = y + ky^3$ as a function of y looks like:



$$f(y) = y + ky^3, k < 0$$

From this graph one can see that if the solution starts close to $y_2 = \sqrt{\frac{1}{-k}}$ but below y_2 then the solution is going to have positive derivative. As a result it is going to increase as long as it is less than y_2. In fact, $y(x)$, is not going to reach the value y_2 because that is going to contradict the uniqueness theorem. Hence $y(x)$ is going to have a limit, say L. One can show that in this case the solution is defined for all $x \in [0, \infty)$, So, we can let x go to infinity in the original DE and obtain that $\lim_{x\to\infty} y' = L + kL^3$. One can show that $\lim_{x\to\infty} y' = 0$. This implies that $L = y_2$.

Remark: The point $k = 0$ is called a **bifurcation** point. By definition, a value of a parameter k is a bifurcation point if the behavior of the critical points (solutions of $f(y, k) = 0$) changes as k increases. The graph of the points (k, c) with $f(c, k) = 0$ is called **bifurcation diagram**.

In some other texts, the a critical point which is stable is also called a **sink**. If the derivative of f exists at such a point then one checks if $f'(c) < 0$ and concludes that the critical point is a sink or stable. If $f'(c) > 0$ then one sees that such a point is not stable or sometime called **source**. If $f'(c) = 0$ or $f'(c)$ doesn't exist, the critical point is said to be a **node**.

Some useful ingredients here are:

Problem 2.1.4. *Let g be differentiable on $[0, \infty)$ such that $\lim\limits_{x\to\infty} g(x)$ and $\lim\limits_{x\to\infty} g'(x)$ exist. Show that $\lim\limits_{x\to\infty} g'(x) = 0$.*

Problem 2.1.5. *Suppose that f is some differentiable function on (a, b) with $c \in (a, b)$ a critical point ($f(c) = 0$) such that $f(y) > 0$ if $y < c$ and $f(y) < 0$ for $y > c$. Show that the initial value problem*

$$\begin{cases} y' = f(y) \\ y(0) = y_0 \in (a, b) \end{cases} \tag{2.4}$$

has a unique solution $y(x)$ defined for all $x \geq 0$ and $\lim\limits_{x\to\infty} y(x) = c$.

Problem 2.1.6. *Let $k > 0$ and f be a differentiable function defined on $[0, \infty)$ such that $\lim\limits_{x\to\infty} [f'(x) + kf(x)] = L$. Show that $\lim\limits_{x\to\infty} f'(x) = 0$.*

Notice that Problem 0.6 generalizes Problem 0.4.

2.1.2 Vertical motion under gravitational force and air resistance proportional to the velocity

A simple application of this analysis can be done for the case of movement of a body with mass m near the surface of the earth subject to gravitation and friction to the air. If one assumes that the friction force is $F = kv$ and opposed to the direction of the movement all the time we get the DE:

$$m\frac{dv}{dt} = -kv + mg$$

or

$$\frac{dv}{dt} = -\rho v + g. \tag{2.5}$$

One can easily see that $v_l = \frac{\rho}{g} = \frac{mg}{k}$ is a stable solution of this differential equation. This speed is called the **terminal speed**. Please read the analysis done in the book for the case the friction is proportional to the square of the velocity. In this case the terminal speed is $\sqrt{\frac{\rho}{g}}$.

2.1.3 Euler's method of approximating the solution of a first-order DE

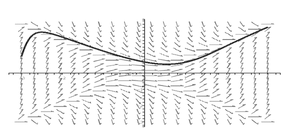
Algorithm: Given the initial value problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \tag{2.6}$$

Euler's method with step size h consists in using the recurrent formula $y_{k+1} = y_k + hf(x_0 + kh, y_k)$ for $k = 0, 1, 2, ..., n$ in order to compute the approximation y_n of the solution of (2.6) at $x = x_n$. The difference $|y(x_n) - y_n|$ is called the **cumulative error**.

The figure below has been obtained with Maple using Euler's method with step size $h = 0.001$, $n = 6000$, for the initial value problem

$$\begin{cases} y' = x^2 - y^2 \\ y(-3) = 1. \end{cases} \tag{2.7}$$



Euler's Method, h=0.001,n=6000

The exact solution of this DE is difficult to calculate but there is an expression of it in terms of Bessel functions. Maple 9 doesn't handle it properly so we cannot compute the cumulative error. There is a theorem that gives some information about the cumulative error.

Theorem 2.1.7. *Assume that the function f in (2.6) is continuous and differentiable on some rectangle $\mathcal{R} = [a, b] \times [c, d]$. Then there exist a constant $C > 0$ (independent of h and as a result, independent of n) such that $|y(x_n) - y_n| < Ch$ as long as $x_n \in (a, b)$, where y_n is computed with the Euler's method with step size h.*

This constant C depends only on the function f and on the rectangle $\mathcal{R}$. Theoretically this implies that by taking h small enough we can get any accuracy we want for the solution.

One can obtain better approximations if one uses the *improved Euler's approximation method* or Runge-Kutta method (please see the book).

Homework: For the first test work problems at the first chapter review on page 76.

Section 2.2 pages 96–97: 1-12, 21, 22;

2.2 Lecture VI

Quotation: *"If there is a problem you can't solve, then there is an easier problem you can solve: find it." George Pólya*

Second-order linear DE, principle of superposition for linear homogeneous equations, existence and uniqueness for linear DE, initial value problem for second-order DE, linear independence of two functions, Wronskian, general solution of linear second-order homogeneous DE, constant coefficients, characteristic equation, the case of real roots, the case of repeated roots and the case of pure complex roots.

The type of equations we are going to be concerned with are DE that could be reduced to

$$y'' + p(x)y' + q(x)y = f(x), \tag{2.8}$$

for some continuous functions p, q, f on an open interval I. Recall that if $f = 0$ then we called the DE homogeneous.

An important property for homogenous linear equations is the following:

Theorem 2.2.1. (Superposition principle) *If y_1 and y_2 are two solutions of $y'' + p(x)y' + q(x)y = 0$, then $z(x) = c_1y_1(x) + c_2y_2(x)$ is also a solution for every constants c_1 and c_2.*

Proof. Because $y_1'' + p(x)y_1' + q(x)y_1 = 0$ and $y_2'' + p(x)y_2' + q(x)y_2 = 0$, we can multiply the first equation by c_1 and the second by c_2 and then add the two new equations together. Then we obtain $z'' + p(x)z' + q(x)z = 0$. ∎

The existence and uniqueness theorem takes a special form in this case.

Theorem 2.2.2. (Existence and Uniqueness) *For the initial value problem*

$$\begin{cases} y'' + p(x)y + q(x)y = f(x) \\ y(a) = b_1, \\ y'(a) = b_2 \end{cases} \tag{2.9}$$

assume that p, q and f are continuous on an interval I containing a. Then (2.9) has a unique solution on I.

The problem (2.9) is called an **initial value problem** associated to a second-order DE.

Example: Suppose we take the differential equation in Problem 16, page 156: $y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 0$ and the initial condition $y(1) = 3$ and $y'(1) = 2$. Then by applying Theorem 0.2 we know that this initial value problem should have a solution defined on $(0, \infty)$. If we take the the two solutions given in the problem $y_1 = \cos(\ln x)$ and $y_2 = \sin(\ln x)$ we can use the Superposition Principle to find our solution by determining the constants c_1 and c_2 from the system:

$$\begin{cases} c_1y_(1) + c_2y_2(1) = 3 \\ c_1y'_1(1) + c_2y'_2(1) = 2 \end{cases} \tag{2.10}$$

or $c_1 = 3$, $c_2 = 2$. This gives $y(x) = 3\cos(\ln x) + 2\sin(\ln x)$ which exists on $(0, \infty)$, the largest interval on which $p(x) = \frac{1}{x}$ and $q(x) = \frac{1}{x^2}$ are defined and continuous.

Definition 2.2.3. *Two functions f, g defined on an interval I are said to be linearly independent on I, if $c_1f(x) + c_2g(x) = 0$ for all $x \in I$ implies $c_1 = c_2 = 0$.*

If two functions are not linearly independent on I, they are called **linearly dependent** on I.

Example I: Suppose $f(x) = \cos 2x$ and $g(x) = \cos^2 x - \frac{1}{2}$ and $I = \mathbb{R}$. These two functions are linearly dependent on $\mathbb{R}$ since $f(x) + (-2)g(x) = 0$ for all $x \in \mathbb{R}$.

Example II: Let us take $f(x) = x$ and $g(x) = |x|$ and $x \in I = \mathbb{R}$. These two are linearly independent since $C_1 f(x) + C_2 g(x) = 0$ for all $x \in \mathbb{R}$. This implies $C_1 + C_2 = 0$ if $x = 1$ for instance $x = 1$ and $-C_1 + C_2 = 0$ if $x = -1$. This attracts $C_1 = C_2 = 0$ which means that f and g are linearly independent.

Definition 2.2.4. *For two differentiable functions f and g on I, the* **Wronskian** *of f and g is the determinant*

$$W(f,g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - f'(x)g(x), \quad x \in I.$$

The next theorem characterizes solutions of second-order DE which are linearly independent.

Theorem 2.2.5. *Let y_1 and y_2 be two solutions of $y'' + p(x)y' + q(x)y = 0$ defined on open interval I, where p and q are continuous. Then y_1 and y_2 are linearly independent if and only if $W(y_1, y_2)(x) \neq 0$ for all $x \in I$.*

Proof. ($\Longleftarrow$) Let us assume that $W(y_1, y_2)(x) \neq 0$ for all $x \in I$. By way of contradiction if the solutions y_1 and y_2, are linearly dependent then $y_1 = cy_2$ for some constant c. Then $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = cy_2 y_2' - (cy_2)' y_2 = 0$. So, if $W(y_1, y_2)(x) \neq 0$ for all $x \in I$. This contradiction shows that the two solutions must be linearly independent.

($\Longrightarrow$) Assume that y_1 and y_2 are linearly independent. This means that $c_1 y_1 + c_2 y_2 = 0$ on I implies $c_1 = c_2 = 0$. We will follow the idea from the Problem 32, page 156. Since $y_1'' + p(x)y_1' + q(x)y_1 = 0$ and $y_2'' + p(x)y_2' + q(x)y_2 = 0$ we can multiply the first equation by y_2 and the second by y_1 and subtract them. We get $y_1'' y_2 - y_1 y_2'' + p(x)(y_1 y_2' - y_2 y_1') = 0$. In a different notation $W'(x) = p(x)W(x)$. This equation in W is linear with the solution $W(x) = W_0 e^{\int p(x)dx}$. This implies that $W(x) \neq 0$ if $W_0 \neq 0$. So, we are done if $W_0 \neq 0$. Again by way of contradiction let us assume that $W_0 = 0$. Then $W(x) = 0$ for all $x \in I$. Hence $(y_1/y_2)' = 0$ which means $y_1/y_2 = c$ or $y_1(x) - cy_2(x) = 0$ for some nonzero constant c and for all $x \in I$ where $y_2(x) \neq 0$. If $y_2(t) = 0$, for some t, then as a corollary of Theorem 2.9 we cannot have $y_2'(t) = 0$ because that will attract $y_2 \equiv 0$. Therefore $W(t) = 0$ implies $y_1(t) = 0$ which means $y_1(x) - cy_2(x) = 0$ for al $x \in I$ and this contradicts the assumption on y_1 and y_2 as being linear independent. It remains that $W(x) \neq 0$ for all $x \in I$. ∎

The next theorem tells us how the general solution of a homogeneous second-order linear differential equation looks like.

Theorem 2.2.6. *If y_1 and y_2 are two linearly independent solutions of $y''+p(x)y'+q(x)y=0$ defined on open interval I, where p and q are continuous then any solution y can be written as $y=c_1y_2+c_2y_2$.*

Proof. Let us start with an arbitrary solution y. Consider an arbitrary point $x_0 \in I$. From the previous theorem we see that $W(x_0) \neq 0$. Hence by Crammer's rule the system

$$\begin{cases} c_1y_1(x_0)+c_2y_2(x_0)=y(x_0) \\ c_1y_1'(x_0)+c_2y_2'(x_0)=y'(x_0) \end{cases}$$

has a unique solution in c_1 and c_2. Hence y and $z=c_1y_2+c_2y_2$ both satisfy the initial value problem

$$\begin{cases} w''+p(x)w+q(x)w=0 \\ w(x_0)=y_0, \\ w'(x_0)=y'(x_0). \end{cases} \tag{2.11}$$

Using the uniqueness property of the solution (Theorem 2.9) we see that the two solutions must coincide: $y=c_1y_1+c_2y_2$. ■

Two linearly independent solutions of a second-order linear homogeneous DE are called a **fundamental set of solutions** for this DE.

2.2.1 Linear second-order DE with constant coefficients

If the DE is of the form $ay''+by'+cy=0$ we can find two solutions which are linearly independent by going first to the **characteristic equation**:

$$ar^2+br+c=0 \tag{2.12}$$

Theorem 2.2.7. *(a) If the roots of the equation (2.12) are real, say r_1 and r_2, and distinct then two linearly independent solutions of $ay''+by'+cy=0$ are e^{r_1x} and e^{r_2x}. The general solution of the DE is then given by $\boxed{y(x)=c_1e^{r_1x}+c_2e^{r_2x}}$.*

(b) If the roots are real but $r_1=r_2=r$ then two linearly independent solutions of $ay''+by'+cy=0$ are e^{rx} and xe^{rx}. The general solution of the DE is then given by $\boxed{y(x)=(c_1+c_2x)e^{r_1x}}$.

(c) If the two roots are pure imaginary ones, say $r_{1,2}=\alpha+i\beta$ then two linearly independent solutions of $ay''+by'+cy=0$ are $e^{\alpha x}\sin\beta x$ and $e^{\alpha x}\cos\beta x$. The general solution of the DE is then given by $\boxed{y(x)=(c_1\sin\beta x+c_2\cos\beta x)e^{\alpha x}}$.

PROOF. We need to check in each case that the given pair of functions form a fundamental set of solutions.

Case (a) The functions $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$, $x \in \mathbb{R}$, satisfy the DE, $ay'' + by' + cy = 0$, because r_1 and r_2 satisfy the characteristic equation (2.12). The Wronskian of these two functions is $W(y_1, y_2)(x) = (r_2 - r_1)e^{(r_1+r_2)x} \neq 0$ since we assume in this case $r_1 \neq r_2$.

Case (b) The two functions this time are $y_1(x) = e^{r_1 x}$ and $y_2(x) = xe^{r_1 x}$. The only novelty here is why y_2 must be a solution: $y_2'(x) = (r_1 x + 1)e^{r_1 x}$, $y_2''(x) = (r_1^2 x + 2r_1)e^{r_1 x}$ and $ay_2'' + by_2' + c = \left[(ar_1^2 + br_1 + c)x + 2ar_1 + b)\right] e^{r_1 x} \equiv 0$ since $r_1 = r_2 = \frac{-b}{2a}$. We have $W(y_1, y_2)(x) = e^{r_1 x} \neq 0$ for all $x \in \mathbb{R}$.

Case (c) Here $y_1(x) = e^{\alpha x} \sin \beta x$ and $y_2(x) = e^{\alpha x} \cos \beta x$. If we calculate $y_1'(x) = (\alpha \sin \beta x + \beta \cos \beta x)e^{\alpha x}$ and $y_1''(x) = \left[(\alpha^2 - \beta^2) \sin \beta x + 2\alpha\beta \cos \beta x\right] e^{\alpha x}$, and $ay_1'' + by_1 + cy_1 = \left[(a(\alpha^2 - \beta^2) + b\alpha + c) \sin \beta x + (2a\alpha\beta + b\beta) \cos \beta x\right] e^{\alpha x}$. But we know from the quadratic formula that $\alpha = \frac{-b}{2a}$ and $\beta = \frac{\sqrt{4ac-b^2}}{2a}$. Hence $a(\alpha^2 - \beta^2) + b\alpha + c = 0$ and $2a\alpha + b = 0$ which in turn implies $ay_1'' + by_1 + c \equiv 0$. Similarly one can check that $ay_2'' + by_2 + cy_2 \equiv 0$. The Wronskian is $W(y_1, y_2)(x) = -2\beta e^{\alpha x} \neq 0$ for all $x \in \mathbb{R}$ (in this case $\beta \neq 0$). ■

Examples: Problem 34, page 156. The DE is $y" + 2y' - 15y = 0$ whose characteristic equation is $r^2 + 2r - 15 = 0$. This has two real solutions $r_1 = -5$ and $r_2 = 3$. Hence the general solution of this equation is $y(x) = c_1 e^{3x} + c_2 e^{-5x}$, $x \in \mathbb{R}$.

In Problem 40, page 156 the DE is $9y'' - 12y' + 4y = 0$. The characteristic equation is $9r^2 - 12r + 4 = 0$ whose solutions are $r_1 = r_2 = \frac{2}{3}$. Hence the general solution of this DE is $y(x) = (c_1 x + c_2)e^{2x/3}$, $x \in \mathbb{R}$.

If the DE is $y'' + 4y' + 13y = 0$ then the characteristic equation $r^2 + 4r + 13 = 0$ has pure complex roots $r_{1,2} = -2 \pm 3i$. Therefore the general solution of the given differential equation is

$$y(x) = (c_1 \sin 3x + c_2 \cos 3x)e^{-2x}.$$

Homework:

Section 3.1 pages 156–157: 13-16, 24-26, 31-42, 51;

2.3 Lecture VII

Quotation: *"If a nonnegative quantity was so small that it is smaller* *than any given one, then it certainly could not be anything but zero. To those who ask what the infinitely small quantity in mathematics is, we answer that it is actually zero. Hence there are not so many mysteries hidden in this concept as they are usually believed to be. These supposed mysteries have rendered the calculus of the infinitely small quite suspect to many people. Those doubts that remain we shall thoroughly remove in the following pages, where we shall explain this calculus. " Leonhard Euler*

Superposition Principle for n-order linear homogeneous DE; Existence and uniqueness for n-order linear DE; Linearly independent and linearly dependent set of functions; Wronskian of a set of n, $(n-1)$-times differentiable functions; Characterization theorem of independent solutions; General solutions of an n-order homogeneous linear DE; Complementary solution y_c and particular solution y_p of an n-order linear DE, Fundamental set of solutions of an n-order homogeneous linear DE; General solutions of an n-order linear DE, n-order linear homogeneous DE with constant coefficients

This lecture is basically a generalization of the previous one. Let us fix n a natural number greater or equal to 2. We are assuming that the n-order linear DE has been reduced to the form:

$$y^{(n)} + p_1(x)y^{(n-1)} + ... + p_n(x)y = f(x), \tag{2.13}$$

where p_1, p_2, ..., p_n, f are continuous on an open interval I. The homogeneous DE associated to (4.5) is

$$y^{(n)} + p_1(x)y^{(n-1)} + ... + p_n(x)y = 0. \tag{2.14}$$

As before an important property for the homogeneous case is the principle of superposition:

Theorem 2.3.1. (Superposition Principle) *If y_k, $k = 1...n$ are solutions of then (2.14) then the function $z(x) = \sum_{k=1}^{n} c_k y_k(x)$, $x \in I$ is also a solution of (2.14) for every value of the constants c_k, $k = 1...n$.*

The proof of this is following exactly the same steps as in the case $n = 2$. ■

The existence and uniqueness theorem needs to formulated in the following way.

Theorem 2.3.2. (Existence and Uniqueness) *For the initial value problem*

$$\begin{cases} y^{(n)} + p_1(x)y^{(n-1)} + ... + p_n(x)y = f(x) \\ y(a) = b_1, y'(a) = b_2, ..., y^{(n-1)}(a) = b_n, \end{cases} \tag{2.15}$$

assume that $p_1, p_2, ..., p_n, f$ are continuous on an interval I containing a. Then (2.15) has a unique solution on I.

The problem (2.15) is called the **initial value problem** associated to (4.5).

Example: Suppose we take the differential equation in Problem 20, page 168: $x^3y''' + 6x^2y'' + 4xy' - 4y = 0$ and the initial condition $y(1) = 1$, $y'(1) = 5$, $y''(1) = -11$. We can rewrite the equation as $y''' + \frac{6}{x}y'' + \frac{4}{x^2}y' - \frac{4}{x^3}y = 0$.

Then by applying Theorem 2.3.2 we know that this initial value problem should have a solution defined on $(0, \infty)$. If we take the three solutions given in the problem's statement: $y_1 = x$, $y_2 = \frac{1}{x^2}$ and $y_3 = \frac{\ln x}{x^2}$. We can use the Superposition Principle to find our solution by determining the constants c_1, c_2 and c_3 from the system:

$$\begin{cases} c_1y_(1) + c_2y_2(1) + c_3y_3(1) = 1 \\ c_1y'_1(1) + c_2y'_2(1) + c_3y'_3(1) = 5 \\ c_1y''_1(1) + c_2y''_2(1) + c_3y''_3(1) = -11 \end{cases}$$

or

$$\begin{cases} c_1 + c_2 \qquad = 1 \\ c_1 - 2c_2 + c_3 = 5 \\ \qquad 6c_2 - 5c_3 = -11. \end{cases}$$

Solving this system of 3×3 linear equations we get $c_1 = 2$, $c_2 = -1$ and $c_3 = 1$. This gives the unique solution of our initial value problem $y(x) = 2x - \frac{1}{x^2} + \frac{\ln x}{x^2}$ which exists on $I = (0, \infty)$, the largest interval on which p_1, p_2, p_3 are defined and continuous and of course containing the initial value for x ($1 \in I$).

Definition 2.3.3. *A set of functions f_k, $k = 1...n$, defined on an interval I, is said to be linearly independent on I, if $\sum_{k=1}^{n} c_k f_k(x) = 0$ for all $x \in I$ implies $c_1 = c_2 = ... = c_n = 0$.*

If a set of functions is not linearly independent on I, the set is called **linearly dependent** on I. By negation of the above definition we see that a set of n functions is linearly dependent on I if there exist c_k not all zero such that $\sum_{k=1}^{n} c_k f_k(x) = 0$ for all $x \in I$.

Example: Suppose $f_1(x) = \sin x,\ f_2(x) = \sin 3x,\ ...,\ f_n(x) = \sin(2n-1)x$ and $f_{n+1}(x) = \dfrac{(\sin nx)^2}{\sin x}$. These (n+1) functions are linearly dependent on $(0, \pi/2)$ since $f_1 + f_2 + f_3 + ... + f_n - f_{n+1} \equiv 0$ (please check!).

Definition 2.3.4. *For n functions, f_1, ...,f_n, which are $(n-1)$-times differentiable on I, the* **Wronskian** *of f_1, ..., f_n is the function calculated by the following determinant*

$$W(f_1, ..., f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & ... & f_n(x) \\ f_1'(x) & f_2'(x) & ... & f_n'(x) \\ ... & ... & ... & ... \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & ... & f_n^{(n-1)}(x) \end{vmatrix}, \quad x \in I.$$

The next theorem characterizes a set of n solutions of an n-order homogeneous linear DE to form a linearly independent set of functions.

Theorem 2.3.5. *Let y_1, y_2,..., y_n be n-solutions of (2.14). Then y_1, y_2,..., y_n forms a set of linearly independent functions if and only if*

$$W(y_1, y_2, ..., y_n)(x) \neq 0$$

for all $x \in I$.

Proof. ($\Longleftarrow$) For *sufficiency* let us proceed as before (in the case $n = 2$) using an argument by contradiction. If the solutions y_1, y_2,..., y_n are linearly dependent then $\sum_{k=1}^{n} c_k y_k \equiv 0$ for some constants c_k not all zero. Then $W(y_1, y_2, ..., y_n) \equiv 0$ because the determinant has one column is a linear combination of the others. So, if $W(y_1, y_2, ..., y_n)(x) \neq 0$ for all $x \in I$ then the set of solutions must be linearly independent.

($\Longrightarrow$) For *necessity*, let us assume the solutions y_1,y_2,...,y_n are linearly independent and again by way of contradiction suppose that their Wronskian is zero for some point $a \in I$. This means that the following homogeneous linear system of equations in $c_1, c_2, ..., c_n$

$$\begin{cases} c_1 y_1(a) + c_2 y_2(a) + ... + c_n y_n(a) = 0 \\ c_1 y_1'(x) + c_2 y_2'(x) + ... + c_n y_n'(x) = 0 \\ ... \\ c_1 y_1^{(n-1)}(a) + c_2 y_2^{(n-1)}(a) + ... + c_n y_n^{(n-1)}(a) = 0, \end{cases} \tag{2.16}$$

has a non-trivial solution. We take such a non-trivial solution, $c_1, c_2, ..., c_n$, and consider the function

$$z(x) = \sum_{k=1}^{n} c_k y_k(x)$$

which by the Superposition Principle is a solution of (2.14). Since (2.16) can be written as $z(a) = z'(a) = ... = z^{(n-1)}(a) = 0$ we can apply the uniqueness of solution (Theorem 2.3.2) and conclude that $z \equiv 0$. But that contradicts the fact that y_1,y_2,...,y_n are linearly independent. ■

Example: The equation $y^{(n)} = 0$ has n linearly independent solutions: $y_1 = 1$, $y_2(x) = x$, $y_3(x) = x^2$, ..., $y_n = x^{n-1}$ on any given interval I since the Wronskian of these functions is equal to $1!2!...(n-1)!$ for all $x \in I$ (please check !).

The next theorem tells us how the general solution of a homogeneous n-order linear differential equation looks like.

Theorem 2.3.6. *If $y_1, y_2, ..., y_n$ are n linearly independent solutions of (2.14) defined on the open interval I, then any solution z of (2.14) can be written as $z(x) = \sum_{k=1}^{n} c_k y_k(x)$, $x \in I$, for some constants c_k, $k = 1...n$.*

Proof. The proof is the same as in the case $n = 2$. Let us start with an arbitrary solution z. Consider an arbitrary point $a \in I$. From the previous theorem we see that $W(y_1, ..., y_n)(a) \neq 0$. Hence by Crammer's rule the system

$$\begin{cases} c_1y_1(a) + c_2y_2(a) + ... + c_ny_n(a) = z(a) \\ c_1y_1'(x) + c_2y_2'(x) + ... + c_ny_n'(x) = z'(a) \\ ... \\ c_1y_1^{(n-1)}(a) + c_2y_2^{(n-1)}(a) + ... + c_ny_n^{(n-1)}(a) = z^{(n-1)}(a), \end{cases} \tag{2.17}$$

has a unique solution for $c_1, c_2, ..., c_n$. Again if we denote $w(x) = \sum_{k=1}^{n} c_k y_k(x)$, $x \in I$, then w is a solution of (2.14) and satisfies the initial conditions $w(a) = z(a)$, $w'(a) = z'(a)$, ...,$w^{(n-1)}(a) = z^{(n-1)}(a)$. Again by Theorem 2.3.2 there exist only one such solution. Therefore $z \equiv w$. ■

A set of n linearly independent solutions of a n-order linear homogeneous DE is called a **fundamental set of solutions** for this DE. So, in solving such a DE we are looking for a fundamental set of solutions. If the differential equation is not homogeneous we have the following characterization of the general solution.

Theorem 2.3.7. *If $y_1, y_2, ..., y_n$ are n linearly independent solutions of (2.14) defined on the open interval I, and y_p is a particular solution of (4.5), then any solution z*

of (4.5) can be written as

$$\boxed{z(x) = y_p(x) + \sum_{k=1}^{n} c_k y_k(x), \quad x \in I},$$

for some constants c_k, $k = 1...n$.

Proof. If y_p is a particular solution of (4.5) then $z - y_p$ is a solution of (2.14). Hence by Theorem 2.3.6, $z(x) - y_p(x) = \sum_{k=1}^{n} c_k y_k(x)$, $x \in I$, for some constants c_k, $k = 1...n$. ∎

Definition The function $\sum_{k=1}^{n} c_k y_k(x)$ is called a *complementary function* associated to (4.5).

2.3.1 Linear n-order linear DE with constant coefficients

We are going to study the particular situation of (4.5) or (2.14) in which the equation is of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + ... + a_n y = 0. \tag{2.18}$$

where a_k are just constant real numbers.

As in the case $n = 2$ the discussion here is going to be in terms of the solutions of the **characteristic equation**:

$$a_0 r^n + a_1 r^{n-1} + ... + a_n = 0. \tag{2.19}$$

Theorem 2.3.8. *A fundamental set of solutions, $\mathcal{S}$, for (2.18) can be obtained using the following rules*

(a) If a root r of the equation (2.19) is real and has multiplicity k then the contribution of this root to $\mathcal{S}$ is with the functions

$$e^{rx}, xe^{rx}, ..., x^{k-1}e^{rx}.$$

(b) If a root $r = a + ib$ of the equation (2.19) is pure complex (i.e. $b \neq 0$) and has multiplicity k then the contribution of this root to $\mathcal{S}$ is with the functions

$$e^{ax}\cos bx, e^{ax}\sin bx, xe^{ax}\cos bx, xe^{ax}\sin bx, ..., x^{k-1}e^{ax}\cos bx, x^{k-1}e^{ax}\sin bx.$$

A not very difficult proof of this theorem can be given if one uses an unified approach of the cases (a) and (b) and employing complex-valued functions instead of real-valued ones.

Examples: Problem 12, page 180. The differential equation is $y^{(4)}-3y^{(3)}+3y''-y'=0$. The characteristic equation is $r^4-3r^3+3r^2-r=0$. The roots of this equation are $r_1=1$ with multiplicity 3 and $r_2=0$. Hence the general solution of this equation is $y(x)=(c_1+c_2x+c_3x^2)e^x+c_4$

Problem 18, page 180. The differential equation is $y^{(4)}=16y$. The associated characteristic equation is $r^4-16=0$. The roots of this equation are $r_{1,2}=\pm 2$ and $r_{3,4}=\pm 2i$. Therefore the general solution of this DE is $y(x)=c_1e^{2x}+c_2e^{-2x}+c_3\cos 2x+c_4\sin 2x$, $x\in\mathbb{R}$.

Homework:

Section 3.2 pages 168-169, 14-20, 27, 28-30, 32-36, 43, 44;

Section 3.3 pages 180-181, 1-20, 24-26, 30-32, 34-36, 45, 46, 50;

2.4 Lecture VIII

Quotation: *"To divide a cube into two other cubes, a fourth power or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it." Pierre Fermat*

Topics: *Mechanical vibrations (damped, undamped, free, forced, amplitude, circular frequency, phase angle, period, frequency, time lag, critical damping, overdamped, underdamped), nonhomogeneous equations, undetermined coefficients, variation of parameters*

Suppose we have a body of mass m attached at one end to an ordinary spring. Hooke's law says that the spring acts on the body with a force proportional to the displacement from the equilibrium position.

Denote by x this displacement. Then this force is $F_s=-kx$ where k is called the **spring constant**. Also let us assume that at the other end the body is attached to a shock absorber that provides a force that is proportional to the speed of the body: $F_r=-c\frac{dx}{dt}$. The number c is called the **damping constant**. If there is also an external force $F_e=F(t)$ then according to the Newton's law:

$$F = F_s + F_r + F_e = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx(t) = F(t). \tag{2.20}$$

Some terminology here has become classic: if we ignore all friction forces, i.e. $c = 0$ we say we have an **undamped system** and it is called **damped** if $c > 0$. If the exterior force is zero we say the system is **free** and if the exterior forces are present the movement is called **forced motion**.

2.4.1 Free undamped motion

We have basically the equation

$$m\frac{d^2x}{dt^2} + kx(t) = 0. \tag{2.21}$$

which has the general solution

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t,$$

where $\omega_0 = \sqrt{k/m}$ called the **circular frequency**. This can be written as

$$x(t) = C\cos(\omega_0 t - \alpha), \tag{2.22}$$

where $C = \sqrt{A^2 + B^2}$ is called the **amplitude** and

$$\alpha = \begin{cases} arctan(\frac{B}{A}) \quad if \quad A, B > 0, \\ \pi + arctan(\frac{B}{A}) \quad if \quad A < 0, \\ 2\pi + arctan(\frac{B}{A}) \quad if \quad A > 0 \;\; and \;\; B < 0, \\ \pi/2 \quad if \;A = 0 \;\; and \;\; B \geq 0 \\ 3\pi/2 \quad if A = 0 \;\; and \;\; B < 0, \end{cases} \tag{2.23}$$

is called the **phase angle**. The **period** of this **simple harmonic motion** is simply $T = \frac{2\pi}{\omega_0}$. The physical interpretation is the time necessary to complete one full oscillation. The **frequency** is defined as the inverse of T, i.e. $\nu = \frac{1}{T}$, is usually measured in hetzs (Hz) and measures the number of complete cycles per second. The **time lag** is the quantity $\delta = \frac{\alpha}{\omega_0}$ represents how long it takes to reach the first time the amplitude.

2.4.2 Free damped motion

The equation (4.5) becomes:

$$x'' + 2px' + \omega_0^2 x = 0, \tag{2.24}$$

where ω_0 is as before and $p = \frac{c}{2m}$. The characteristic equation has roots $r_{1,2} = -p \pm \sqrt{p^2 - \omega_0^2}$. As we have seen this leads to a discursion in terms of the discriminant of the equation $p^2 - \omega_0^2 = \frac{c^2 - 4mk}{4m^2}$. We have a critical damping coefficient for $c_{cr} = 2\sqrt{km}$. If $c > c_{cr}$ we say the system is **over-damped** in which case $x(t) \to 0$ as $t \to \infty$ since the general solution is

$$\boxed{x(t) = Ae^{r_1 t} + Be^{r_2 t},\ t \in \mathbb{R}.}$$

There are no oscillations around the equilibrium position and the body passes through the equilibrium position at most once.

If $c = c_{cr}$, the system is **critically-damped** and the general solution is of the form

$$\boxed{x(t) = (A + Bt)e^{rt},\ t \in \mathbb{R}.}$$

and as before $x(t) \to 0$ as $t \to \infty$ and again the body passes through the equilibrium position at most once.

If $c < c_{cr}$ we say the system is **under-damped**. The general solution in this case is

$$\boxed{x(t) = e^{-pt}(A \cos \omega_1 t + B \sin \omega_1 t) = e^{-pt} \cos(\omega_1 t - \alpha),}$$

using the same notations as before. In this case, $\omega_1 = \frac{\sqrt{4mk - c^2}}{2m}$ is called **circular pseudo-frequency**, and $T_1 = \frac{2\pi}{\omega_1}$ is its **pseudo-period**.

2.4.3 Nonhomogeneous linear equations, undetermined coefficients method

In order to determine a particular solution of a nonhomogeneous linear equation of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + ... + a_n y = f(x), \tag{2.25}$$

in special situations one can try a solution of a certain form. This method applies whenever the function f is a finite linear combination of products of *polynomials*, *exponentials*, *cosines* or *sines*. One needs to apply two rules.

Rule 1: If none of the terms of the function f contains solutions of the homogeneous DE associated to (2.25), then the reccomended function to try is y_p a combination of the terms in f and all their derivatives that form a finite set of linearly independent functions.

Example 1: Let us suppose the DE is $y^{(3)}+y = \sin x+e^x$. Since the complementary solution of this equation is $y_c = a_1e^{-x} + [a_2 \cos(x\sqrt{3}/2) + a_3 \sin(x\sqrt{3}/2)]e^{x/2}$ we can try a particular solution to be $y = c_1 \sin x + c_2 \cos x + c_3e^x$. After substituting in the equation we get $(c_2 - c_1) \cos x + (c_2 + c_1) \sin x + 2c_3e^x = \sin x + e^x$. So, it follows that $c_1 = c_2 = c_3 = 1/2$. So, the general solution of the given equation is $y(x) = a_1e^{-x} + [a_2 \cos(x\sqrt{3}/2) + a_3 \sin(x\sqrt{3}/2)]e^{x/2} + \frac{1}{2}(\sin x + \cos x + e^x)$

Example 2: Suppose the DE is $y'' + 2y' - 3y = x^2e^{2x}$. Because the complementary solution of this equation is $y_c = a_1e^x + a_2e^{-3x}$ we can take as a particular solution $y_p = (c_1x^2 + c_2x + c_3)e^{2x}$. Since $y_p' = [2c_1x^2 + (2c_1 + 2c_2)x + c_2 + 2c_3]e^{2x}$ and $y_p'' = [4c_1 + (8c - 1 + 4c_2)x + 2c_1 + 4c_2 + 4c_3)e^{2x}$, we see that $y'' + 2y' - 3y = [5c_1x^2 + (12c_1 + 5c_2)x + 2c_1 + 6c_2 + 5c_3)]e^{2x} = x^2e^{2x}$. Therefore $c_1 = \frac{1}{5}$, $c_2 = \frac{-12}{25}$ and $c_3 = \frac{62}{125}$. Thus the general solution of the given equation is $y(x) = a_1e^x + a_2e^{-3x} + \dfrac{25x^2 - 60x + 62}{125}e^{2x}$.

Rule 2: If the function f contains terms which are solutions of the homogeneous linear DE associated, then one should try as a particular solution, y_p, a linear combination of these terms and their derivatives which are linearly independent multiplied by a power of x, say x^s, where s is the smallest nonnegative integer which makes all the new terms not to be solutions of the homogeneous problem.

Example 3: Let us assume the differential equation we want to solve is $y''+2y'+y = x^2e^{-x}$. So we need to determine the coefficients of the particular solution $y_p = (c_1x^4 + c_2x^3 + c_3x^2)e^{-x}$. After a simple calculation we get $c_1 = \frac{1}{12}$ and $c_2 = c_3 = 0$.

Example 4: Suppose we are given the DE $y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = e^x \sin x$. The complementary solution is $y_c = (a_1 + a_2x + a_3x^2 + a_4x^3)e^x$ we need to look for a particular solution of the form $y_p = (c_1 \sin x + c_2 \cos x)e^x$. If we introduce the differential operator $D = \frac{d}{dx}$ then the equation given is equivalent to $(D - 1)^4y = e^x \sin x$ and we are looking for a particular solution $y_p = u(x)e^x$. Since $(D - 1)y_p = (Du)e^x$ (please check!) we see that $(D - 1)^4y_p = (D^4u)e^x = (c_1 \sin x + c_2 \cos x)e^x$ so $c_1 = 1$ and $c_2 = 0$.

2.4.4 Nonhomogeneous linear equations, Variation of parameters method

We are going to describe the method in the case $n = 2$ but this works in fact for the n-order linear nonhomogeneous DE.

Theorem 2.4.1. *A particular solution of the differential equation $y'' + p(x)y' + q(x)y = f(x)$ is given by*

$$\boxed{y_p = -y_1(x)\int \frac{y_2(x)f(x)}{W(y_1,y_2)(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(y_1,y_2)(x)}dx,} \tag{2.26}$$

where y_1 and y_2 is a fundamental set of solutions of $y'' + p(x)y' + q(x)y = 0$.

Proof. We are looking for a solution of the form $y_p = u_1(x)y_1(x)+u_2(x)y_2(x)$. Differentiating with respect to x we obtain $y_p = u_1'y_1 + u_2'y_2 + u_1y_1' + u_2y_2'$. We are going to make an assumption that is going to simplify the next differentiation:

$$u_1'y_1 + u_2'y_2 = 0. \tag{2.27}$$

Hence, $y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$ and then $y_p'' + p(x)y_p' + q(x)y_p = u_1(y_1'' + p(x)y_1' + q(x)y_1) + u_2(y_2'' + p(x)y_2' + q(x)y_1) + u_1'y_1' + u_2'y_2' = f(x)$. Thus this reduces to

$$u_1'y_1' + u_2'y_2' = f(x). \tag{2.28}$$

Using (2.27) and (2.28) to solve for u_1' and u_2' that gives $u_1'(x) = -\dfrac{y_2(x)f(x)}{W(y_1,y_2)(x)}$ and $u_2'(x) = \dfrac{y_1(x)f(x)}{W(y_1,y_2)(x)}$ which gives (2.26). ■

Example: Let us work Problem 58, page 208. The DE is $x^2y'' - 4xy' + 6y = x^3$. Dividing the equation by x^2 we get $y'' - 4y'/x + 6y/x^2 = x$. So, we have $p(x) = -4/x$ and $q(x) = 6/x^2$ and $f(x) = x$. Two linearly independent solutions are given: $y_1 = x^2$ and $y_2 = x^3$. We have $W(y_1,y_2)(x) = 3x^4 - 2x^4 = x^4 \neq 0$ for $x \in (0,\infty)$. Then $u_1'(x) = -\frac{y_2(x)f(x)}{W(y_1,y_2)(x)} = -1$ which gives $u_1(x) = -x$ and $u_2'(x) = \frac{y_1(x)f(x)}{W(y_1,y_2)(x)} = 1/x$ which implies $u_2(x) = \ln x$. Therefore a particular solution of this equation is $y_p = x^3(\ln x - 1)$ with $x \in (0,\infty)$.

Homework:

Section 3.4 pages 192-193, 1-4, 13, 15, 16, 32, 33;

Section 3.5 pages 207-208, 1-20, 31-40, 47-56, 58-63.

2.5 Lecture IX

Quotation: *"Finally, two days ago, I succeeded - not on account of my hard efforts, but by the grace of the Lord. Like a sudden flash of lightning, the riddle was solved. I am unable to say what was the conducting thread that connected what I previously knew with what made my success possible." Carl Friedrich Gauss*

Topics: *Forced Oscillations, Beats, Resonance, Boundary Value Problems*

2.5.1 Undamped Forced Oscillations

In the previous lecture we studied the mechanical vibrations of a body under the action of a spring, damped forces and exterior forces. The DE was:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx(t) = F(t). \tag{2.29}$$

Now, we assume the exterior force $F(t)$ is of the form $F(t) = F_0 \cos\omega t$ and the damping coefficient $c = 0$. The differential equation that we need to study is of the form

$$mx'' + kx = F_0 \cos\omega t, \tag{2.30}$$

which admits as a complementary solution $x_c(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$, where $\omega_0 = \sqrt{\frac{k}{m}}$. First let us assume that $\omega \neq \omega_0$. Then, to find a particular solution of (4.5) we try $x_p(t) = A\cos\omega t$ using the undetermined coefficient method (no term in $\sin\omega t$ is needed as we can see from the following computation):

$$-Am\omega^2 \cos\omega t + Ak\cos\omega t = F_0 \cos\omega t,$$

which implies $A = \frac{F_0}{k-m\omega^2} = \frac{F_0/m}{\omega_0^2-\omega^2}$. Therefore, the general solution of (4.5) is

$$\boxed{x(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t.}$$

This shows that the solution is a combination of two harmonic oscillations having different frequencies:

$$x(t) = C\cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t. \tag{2.31}$$

where $C = \sqrt{c_1^2 + c_2^2}$ and α is defined by (2.23) but some notation has changed so we update it here:

$$\alpha = \begin{cases} arctan(\frac{c_2}{c_1}) \quad if \quad c_1, c_2 \geq 0, \\ \pi + arctan(\frac{c_2}{c_1}) \quad if \quad c_1 < 0, \\ 2\pi + arctan(\frac{c_2}{c_1}) \quad if \quad c_1 > 0 \quad and \quad c_2 < 0, \pi/2 \quad if \ c_1 = 0 \quad and \quad c_2 \geq 0 \\ 3\pi/2 \quad if c_1 = 0 \quad and \quad c_2 < 0. \end{cases}$$

2.5.2 Beats

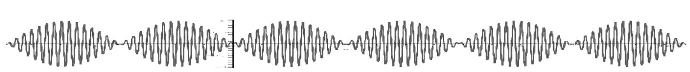
If the amplitude $C = \sqrt{c_1^2 + c_2^2} = \frac{F_0/m}{|\omega_0^2 - \omega^2|}$ and the phase α is zero if $\omega > \omega_0$ or π if $\omega < \omega_0$, which can be accomplished by imposing the initial condition $x(0) = x'(0) = 0$, then the general solution can be written as

$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t)$$

or

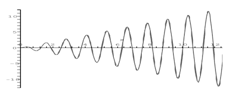
$$\boxed{x(t) = \frac{2F_0/m}{\omega_0^2 - \omega^2}\sin\frac{(\omega_0 - \omega)t}{2}\sin\frac{(\omega_0 + \omega)t}{2}.}$$

If we assume that the two frequencies are close to one another (i.e. $\omega \approx \omega_0$) the expression above explains the behavior of the solution in some sense. We have a product of two harmonic functions, one with a big circular frequency, $(\omega_0 + \omega)/2$, and the other with a smaller one $|\omega_0 - \omega|/2$ which gives the phenomenon of beats. The graph below is the graph of a function of this type: $f(t) = \sin(t)\sin(30t)$ on the interval $t \in [-2\pi, 4\pi]$.



2.5.3 Resonance

Suppose that we have ω getting closer and closer (within any given $\epsilon > 0$) to ω_0. Then $A(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}$ goes to infinity. This is the phenomenon of **resonance**. In fact a particular solution of the problem (4.5) in the case $\omega = \omega_0$ is $x_p(t) = t \sin \omega_0 t$. The graph of the of a function of this type is included below:



This phenomenon is considered to be the explanation of a lot of disasters like the one that happened in 1940 with the Tacoma Narrows Bridge near Seattle. It seeamed like the exterior forces created by the wind created exactly this kind of explosion of the amplitude of the oscillations in the vertical suspension cables. Another classical example is the collapse of the Broughton Bridge near Manchester in England of 1831 when soldiers marched upon it.

2.5.4 Endpoint problems and eigenvalues

We are concerned with second order linear and homogenous DE which have a special type of initial conditions. One such **endpoint** problem is:

$$\begin{cases} y'' + p(x)y' + \lambda q(x)y = 0 \\ a_1 y(a) + a_2 y'(a) = 0 \\ b_1 y(b) + b_2 y'(b) = 0, \end{cases} \tag{2.32}$$

where $a \neq b$. In general only the trivial solution $y \equiv 0$ satisfies (2.32). But for some values of the parameter λ the problem (2.32) may have non-zero solutions. These values are called **eigenvalues** and the corresponding functions are called **eigenfunctions**. A general method to solve (2.32) is to write the general solution of the DE as $y = Ay_1(x, \lambda) + By_2(x, \lambda)$, where y_1 and y_2 is a fundamental set of solutions which is also going to depend of λ. We impose the two initial boundary conditions and rewrite these equations as a system in A and B:

$$\begin{cases} \alpha_1(\lambda)A + \beta_1(\lambda)B = 0 \\ \alpha_2(\lambda)A + \beta_2(\lambda)B = 0 \end{cases} \tag{2.33}$$

This system in A and B has a non-trivial solution if and only if

$$\alpha_1(\lambda)\beta_2(\lambda) - \alpha_2(\lambda)\beta_1(\lambda) = 0. \tag{2.34}$$

One solves this equation and obtains the eigenvalues of (2.32).

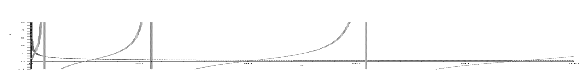
Example: Let us work Problem 6, page 240. The DE is $y'' + \lambda y = 0$ and the boundary conditions are $y'(0) = 0$ and $y(1) + y'(1) = 0$. We are given that all eigenvalues are nonnegative, so we write $\lambda = \alpha^2$.

(a) We have to show that $\lambda = 0$ is not an eigenvalue. If by way of contradiction we assume it is, then some non-zero solution of our problem must exist: $y(x) = A + Bx$, $0 = y'(0) = B$ and then $0 = y(1) + y'(1) = A$ which is a contradiction.

(b) We need to show that the eigenvalues of this problem are the solutions in λ of the equation

$$\tan\sqrt{\lambda} = \frac{1}{\sqrt{\lambda}}.$$

Let $y(x) = A\cos\alpha t + B\sin\alpha t$ be the general solution of our DE without the boundary conditions. Since $y'(0) = 0$ we have $0 = (-\alpha A\sin\alpha t + B\alpha\cos\alpha t)_{|t=0}$ or $B = 0$. Then $y(1) + y'(1) = 0$ implies $A\cos\alpha - A\alpha\sin\alpha = 0$. Since we assume there exit a non-zero solution, we must have $A \neq 0$. Therefore α must satisfy $\cos\alpha - \alpha\sin\alpha = 0$ or $\tan\alpha = \frac{1}{\alpha}$. Corresponding eigenfunctions are $y(x) = A\cos\alpha x$. Since a picture is worth a thousand words let us include the graph of $\lambda \to \tan\sqrt{\lambda}$ and $\lambda \to \frac{1}{\sqrt{\lambda}}$ for $\lambda > 0$.



Homework:

Section 3.6 page 219 Problems 21, 22;

Section 3.8 page 240 Problems 1-6, 13, 14.

Chapter 3

Systems of Differential Equations

3.1 Lecture X

Quotation: *"We [he and Halmos] share a philosophy about linear algebra: we think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury. Paul Halmos: Celebrating 50 Years of Mathematics." Irving Kaplansky*

3.1.1 A non-linear classical example: Kepler's laws of planetary motion

After analyzing observations of Tycho Brache, Johannes Kepler arrived to the following laws of planetary motion:

1. *The orbits of planets are ellipses (with the sun in one of the foci).*

2. *The planets move in such a way on the orbit, that their corresponding ray wipes out an area that varies at a constant rate.*

3. *The square of the planet's period of revolution is proportional to the cube of the major semi-axis of the elliptical orbit.*

We are going to make an assumption here which is not very far from what it happens in the reality (neglect the influence of the planet in question on the sun). The sun contains more than 99% of the mass in the solar system, so the influence of the planets on the sun could be, on a first analysis, neglected. Intuitively it is not hard to believe that the planet X is moving in a fixed plane although this is also a consequence of the movement under the gravitational field. Let us take the origin of the coordinates in this plane centered at the sun.

The position vector corresponding to the planet X is denoted here by $\vec{r} = x(t)\vec{i} + y(t)\vec{j}$ where $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$. The distance between the sun and the planet X is $r = \sqrt{x(t)^2 + y(t)^2}$. According to Newton's law the planet X moves under the action of a force which is inverse proportional to the square of the distance r. The law can be written as a differential equation in the following way:

$$\vec{r}'' = -k\frac{\vec{r}}{r^3}. \tag{3.1}$$

We are going to derive Kepler's laws from (3.1). First let us observe that (3.1) is just the vectorial form of the following second-order non-linear autonomous system of differential equations:

$$\begin{cases} x'' = -k\dfrac{x}{(x^2+y^2)^{3/2}} \\ y'' = -k\dfrac{y}{(x^2+y^2)^{3/2}}. \end{cases} \tag{3.2}$$

It is really a significant fact that this can be reduced to a differential equation that we know how to solve. To see this, let us first introduce polar coordinates, by assuming that the trajectory is written in polar coordinates, $r = r(\theta)$, and we consider two unit vectors that will help us simplify the calculations:

$$\vec{u} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$
and
$$\vec{v} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}.$$

It is easy to check that $\vec{u}\cdot\vec{v} = 0$, and these two vectors clearly depend of time because θ is. Differentiating these two vectors with respect to time we get

$$\begin{gathered} \tfrac{d\vec{u}}{dt} = (-\sin\theta\,\vec{i} + \cos\theta\,\vec{j})\tfrac{d\theta}{dt} = \vec{v}\tfrac{d\theta}{dt} \\ \text{and} \\ \tfrac{d\vec{v}}{dt} = (-\cos\theta\,\vec{i} - \sin\theta\,\vec{j})\tfrac{d\theta}{dt} = -\vec{u}\tfrac{d\theta}{dt}. \end{gathered} \tag{3.3}$$

Since $\vec{r} = r\vec{u}$ after differentiating this equality, we obtain

$$\frac{d\vec{r}}{dt} = r'\vec{u} + r\frac{d\vec{u}}{dt} = r'\vec{u} + r\vec{v}\frac{d\theta}{dt}.$$

Differentiating one more time and using (3.3) we get:

$$\frac{d^2\overrightarrow{r}}{dt^2} = r''\overrightarrow{u} + 2r'\overrightarrow{v}\frac{d\theta}{dt} - r\overrightarrow{u}\left(\frac{d\theta}{dt}\right)^2 + r\overrightarrow{v}\frac{d^2\theta}{dt^2}$$

or

$$\frac{d^2\overrightarrow{r}}{dt^2} = \left(r'' - r\left(\frac{d\theta}{dt}\right)^2\right)\overrightarrow{u} + \left(2r'\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\overrightarrow{v} = -k\frac{\overrightarrow{u}}{r^2}.$$

Identifying the coefficients of $\overrightarrow{u}$ and $\overrightarrow{v}$ in the above relation we obtain

$$\begin{cases} r'' - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{k}{r^2} \\ 2r'\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0. \end{cases} \tag{3.4}$$

The second relation in (3.4) is equivalent to $\frac{d}{dt}(r^2\frac{d\theta}{dt}) = 0$ $(r \neq 0)$. This means $r^2\frac{d\theta}{dt} = h$ for some constant h. This proves the second Kepler's law since

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \left[\lim_{\Delta\theta\to 0}\frac{r(\theta+\Delta\theta)r(\theta)\sin(\Delta\theta)}{2\Delta\theta}\right]\frac{d\theta}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{h}{2}.$$

The first equation in (3.4) can be transformed using the substitution $r = \dfrac{1}{z}$ and changing the independent variable to θ instead of t: $\dfrac{dr}{dt} = -\dfrac{1}{z^2}\dfrac{dz}{d\theta}\dfrac{d\theta}{dt} = -h\dfrac{dz}{d\theta}$ and then $\dfrac{d^2r}{dt^2} = -h\dfrac{d^2z}{d\theta^2}\dfrac{d\theta}{dt} = -h^2z^2\dfrac{d^2z}{d\theta^2}$ which gives

$$-h^2z^2\frac{d^2z}{d\theta^2} - \frac{1}{z}h^2z^4 = -kz^2.$$

Equivalently, this can be written as

$$\frac{d^2z}{d\theta^2} + z = \frac{k}{h^2}. \tag{3.5}$$

As we have seen the general solution of this is $z = A\cos\theta + B\sin\theta + \frac{k}{h^2} = \frac{k}{h^2}(1 + e\cos(\theta-\alpha))$ where $e = \frac{h^2}{k}\sqrt{A^2+B^2}$, $\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}$ and $\sin\alpha = \frac{B}{\sqrt{A^2+B^2}}$. This gives

$$\boxed{r = \frac{L}{1 + e\cos(\theta-\alpha)}.} \tag{3.6}$$

which represents an ellipse if the **eccentricity** e satisfies $0 \leq e < 1$, a parabola if $e = 1$ or a hyperbola if $e > 1$. Since the orbits of the planets are bounded it must be the case that $e < 1$. Comets, by definition, are having parabolic or hyperbolic orbits. [So, according to this definition, Halley's comet is actually not a comet.] This proves the first Kepler's law.

To derive the third Kepler's law, let us integrate the area formula $\frac{dA}{dt} = h/2$ over the interval $[0, T]$, where T is the period of the orbit. Then we get $hT/2 = Area(\text{Ellipse})$, but the area of an ellipse is equal to πab, where a and b are the two semiaxes. The big axis is $a = (\frac{L}{1+e} + \frac{L}{1-e})/2 = \frac{L}{1-e^2}$ and $b = \frac{L}{\sqrt{1-e^2}}$. This means that $\frac{h^2T^2}{4} = \frac{\pi^2L^4}{(1-e^2)^3}$. From here we see that

$$T^2 = \frac{4\pi^2 L}{h^2}a^3 = \frac{4\pi^2}{k}a^3$$

or

$$\boxed{\frac{T^2}{a^3} = \frac{4\pi}{k} = constant,}$$

which is the third Kepler's law.

3.1.2 Linear systems of differential equations

Our general setting here is going to be

$$x'(t) = P(t)x(t) + f(t) \tag{3.7}$$

where $P(t)$ is a $n \times n$ matrix whose coefficients are continuous functions on an interval I, $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^t$ is the column vector of the unknown functions and $f(t) = [f_1(t), f_2(t), ..., f_n(t)]^t$ is a vector-valued function assumed continuous on I as well.

Theorem 3.1.1. *(Existence and uniqueness) The initial value problem*

$$\begin{cases} x'(t) = P(t)x(t) + f(t) \\ x_1(a) = b_1, x_2(a) = b_2, ..., x_n(a) = b_n \end{cases} \tag{3.8}$$

with $a \in I$ has a unique solution defined on I.

We notice here that the number of initial conditions in (4.2) is equal to the number of unknowns.

Theorem 3.1.2. *(**Principle of superposition**) If y_1, y_2, ..., y_n are solutions of the homogeneous problem associated to (3.7), i.e. $x'(t) = P(t)x(t)$, then the vector function $y = \sum_{k=1}^{n} c_k y_k$ is also a solution for every values of the constants c_k.*

Definition 3.1.3. *As before, we say that a set of vector-valued functions, $\{f_1, f_2, ..., f_k\}$, is called* **linearly independent** *if $\sum_{i=1}^{k} c_i f_k(t) = 0$ for all $t \in I$, implies c_i=0 for all $i = 1...k$.*

For a set of n vector valued functions, y_1,...,y_n, the **Wronskian** in this case, $W(y_1, ..., y_n)$, is constructed as the determinant of the matrix $(y_{jk})_{j,k=1..n}$ where $y_j = [y_{1j}, y_{2j}, ..., y_{nj}]^t$. We have a similar characterization of linear independence.

Theorem 3.1.4. *If the system $x'(t) = P(t)x(t)$ admits y_1, ..., y_n as solutions then these are linearly independent if and only if the Wronskian asssociated to them, $W(y_1, ..., y_n)(t)$, is not zero for all $t \in I$.*

The proof of this theorem goes the same way as the one for the similar theorem we studied in the case of n-order linear differential equations. Its proof is based on the existence and uniqueness theorem. Based on this, let us remark that every homogeneous system $x' = Px$ admits a **fundamental set of solutions**, i.e. a set of n linearly independent solutions.

Indeed, one has to use the existence and uniqueness theorem and denote by y_k, the solution of the initial value problem

$$\begin{cases} x'(t) = P(t)x(t) \\ x_1(a) = 0, ..., x_k(a) = 1, ...x_n(a) = 0 \quad (a \in I). \end{cases}$$

Then the Wronskian of the solutions $\{y_1, y_2, ..., y_n\}$ at a is equal to the determinant of the identity matrix, which is, in particular, not zero. By Theorem 3.1.4 we then see that this set of solutions must be linearly independent.

Theorem 3.1.5. *Suppose $\{y_1, y_2, ..., y_n\}$ is a fundamental system of solutions of the system $x'(t) = P(t)x(t)$. Then every other solution of this system, z, can be written as $z(t) = \sum_{k=1}^{n} c_k y_k(t)$, $t \in I$, for some parameters c_k.*

For the nonlinear we have the following theorem:

Theorem 3.1.6. *Suppose* $\{y_1, y_2, ..., y_n\}$ *is a fundamental system of solutions of the system* $x'(t) = P(t)x(t)$ *and* y_p *is a particular solution of* $x'(t) = P(t)x(t) + f(t)$ *Then every other solution of* $x'(t) = P(t)x(t) + f(t)$, *say* z, *can be written as* $z(t) = y_p + \sum_{k=1}^{n} c_k y_k(t)$, $t \in I$, *for some choice of the parameters* c_k.

Homework:

Section 3.6 page 219 Problems 21, 22;

Section 3.8 page 240 Problems 1-6, 13, 14.

3.2 Lecture XI

> **Quotation:** *"The reader will find no figures in this work. The methods which I set forth do not require either constructions or geometrical or mechanical reasonings: but only algebraic operations, subject to a regular and uniform rule of procedure." Joseph-Louis Lagrange , Preface to Mecanique Analytique.*

3.2.1 The eigenvalue method for homogeneous with constant coefficients

In this subsection we assume that the matrix-valued $P(t)$ is a constant function: $A = (a_{j,k})_{j,k=1..n}$. We are going to consider the homogeneous problem

$$x'(t) = Ax(t). \tag{3.9}$$

We remind the reader the definition of an eigenvalue and eigenvector for a matrix A.

Definition 3.2.1. *A complex number* λ *is called an* **eigenvalue** *for the matrix* A *if there exist a nonzero vector* u *such that* $Au = \lambda u$.

It turns out that the eigenvalues of a matrix are the zeros of its characteristic polynomial $p(\lambda) = det(A - \lambda I)$. We have the following simple theorem:

Theorem 3.2.2. *(a) Suppose* λ *is a real eigenvalue of* A *with a corresponding eigenvector* u. *Then the vector-valued function* $v(t) = e^{\lambda t}u$ *is a solution of the DE (3.9).*

(b) If $\lambda = p + iq$ *and the corresponding eigenvector is* $u = a + ib$ *the* $v_1(t) = e^{pt}(a \cos qt - b \sin qt)$, $v_2 = e^{pt}(b \cos qt + a \sin qt)$ *are solutions of the the DE (3.9).*

PROOF. (a) Since $\frac{dv}{dt} = \lambda e^{\lambda t}u$ and $Av(t) = e^{\lambda t}Au = \lambda e^{\lambda t}u$ we see that the function v satisfies (3.9).

(b) Because λ is a solution of the characteristic polynomial whose coefficients are real, the complex conjugate of λ, $\overline{\lambda}$, must also be an eigenvalue. In fact, the complex conjugate of u, $\overline{u} = a - ib$, is the corresponding eigenvector of $\overline{\lambda}$. Because $w_1 = e^{\lambda t}u$ is a solution of (3.9), as we have seen above, and then $w_2 = e^{\overline{\lambda} t}\overline{u}$ is also a solution. Therefore, by the superposition principle, any linear combination of these is a solution also. But then, we are done, since a simple calculation shows that $v_1 = (w_1 + w_2)/2$ and $v_2 = -i(w_1 - w_2)/2$. ∎

Theorem 3.2.3. *Suppose the matrix A has n different solutions $\lambda_1, ..., \lambda_k$, $\lambda_{k+1}, \overline{\lambda_{k+1}}, ...$ where λ_j are real for $j = 1...k$ and pure complex for the rest of them. We denote by u_j the corresponding eigenvectors of the eigenvalue λ_j. Then a fundamental system of solutions of (3.9) can be given by $\{v_1, ..., v_n\}$ where $v_j = e^{\lambda t}u_j$ for $j = 1...k$ and $v_j = e^{p_j t}(a_j \cos qt - b_j \sin qt)$, $v_{j+1} = e^{p_j t}(b_j \cos qt + a_j \sin qt)$,... for $j \geq k+1$, where $\lambda_j = p + iq$ and $u_j = a_j + ib_j$ for $j \geq k+1$.*

PROOF. Let us show that the system is linearly independent. First, let us assume that all of the vector value functions are of the form $v_j = e^{\lambda t}u_j$. We observe that u_1, ..., u_n are linearly independent as vectors in $\mathbb{R}^n$. Indeed, this is happening because if $\sum_{j=1}^n c_j u_j = 0$ then, applying A several times to this equality, we get $\sum_{j=1}^n {\lambda_j}^s c_j u_j = 0$. This implies $c_j u_{jl} = 0$ for every $l = 1..n$ because the main the determinant of the homogeneous linear system obtained is a Vandermonde determinant. The Vandermonde determinant is equal to $\prod\limits_{j<l}(\lambda_j - \lambda_l) \neq 0$. Because each u_j is not zero, there exist a component u_{jl} which is not zero. Then we get $c_j = 0$. Hence u_1, ..., u_n are linearly independent.

Using this we get that $det([u_1, ..., u_n]) \neq 0$. But then the Wronskian of $v_1, ..., v_n$ is $e^{(\lambda_1 + ... + \lambda_n)t}det([u_1, ..., u_n]) \neq 0$. Using Theorem 3.1.4 we see that $v_1, ..., v_n$ are linearly independent.

The case when we have some pure complex eigenvalues, using elementary properties of determinants, we obtain that the Wronskian value for the given functions $v_1, ..., v_n$ change from the value calculated above for functions of the type $e^{\lambda_j t}u_j$ just by a multiple of a power of 2. ∎

Example: Let's take the Problem 20, page 312:

$$\begin{cases} x_1' = 5x_2 + x_2 + 3x_3 \\ x_2' = x_1 + 7x_2 + x_3 \\ x_3' = 3x_1 + x_2 + 5x_3. \end{cases}$$

Here the matrix is $A = \begin{pmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{pmatrix}$. The characteristic polynomial is $p(\lambda) = det(A-\lambda)$. Let us notice that

$$det\begin{pmatrix} 5-\lambda & 1 & 3 \\ 1 & 7-\lambda & 1 \\ 3 & 1 & 5-\lambda \end{pmatrix} = det\begin{pmatrix} 9-\lambda & 9-\lambda & 9-\lambda \\ 1 & 7-\lambda & 1 \\ 3 & 1 & 5-\lambda \end{pmatrix} =$$
$$(9-\lambda)det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 7-\lambda & 1 \\ 3 & 1 & 5-\lambda \end{pmatrix} = (9-\lambda)det\begin{pmatrix} 1 & 1 & 1 \\ 0 & 6-\lambda & 0 \\ 0 & -2 & 2-\lambda \end{pmatrix} =$$
$$(9-\lambda)(6-\lambda)(2-\lambda).$$

Hence $\lambda_1 = 9$, $\lambda_2 = 2$, and $\lambda_3 = 6$. Three eigenvectors corresponding to these are $u_1 = [1,1,1]^t$, $u_2 = [-1,0,1]^t$, $u_3 = [1,-2,1]^t$. Then the general solution of this system is

$$\begin{cases} x_1 = c_1e^{9t} - c_2e^{2t} + c_3e^{6t} \\ x_2 = c_1e^{9t} - 2c_3e^{6t} \\ x_3 = c_1e^{9t} + c_2e^{2t} + c_3e^{6t}. \end{cases}$$

Homework:

Section 5.1 pages 297-298 Problems 1-30;

Section 5.2 page 312 problem 1-26;

3.3 Lexture XII

> **Quotation:** *"[Regarding $\sqrt{-1}$ or what we denote these days by i, the building block of imaginary complex number system]: ... we can repudiate completely and which we can abandon without regret because one does not know what this pretended sign signifies nor what sense one ought to attribute to it." Augustin-Louis Cauchy said in 1847.*

3.3.1 Multiplicity vs Dimension of the Eigenspace

In this part we assume that the system

$$x'(t) = Ax(t). \tag{3.10}$$

has constant coefficients: $A = (a_{j,k})_{j,k=1..n}$, $a_{j,k} \in \mathbb{R}$.

Definition 3.3.1. *We say that an eigenvalue λ_0 has multiplicity m if the characteristic polynomial $p(\lambda) = det(A-\lambda I) = (\lambda-\lambda_0)^m q(\lambda)$ with $q(\lambda_0) \neq 0$. The multiplicity of λ_0 will be denoted by $m(\lambda_0)$.*

Definition 3.3.2. *The set of vectors $E_{\lambda_0} := \{v : \ Av = \lambda_0 v \ \}$ is a linear subspace invariant to A and the dimension of it is called the dimension of the eigenspace associated to λ_0 denoted $de(\lambda_0)$.*

In general we have the following relationship between these numbers:

Theorem 3.3.3. *If λ is an eigenvalue of A we have $de(\lambda) \leq m(\lambda)$.*

In the case $m(\lambda) = de(\lambda)$ we call λ **complete**. As we have seen before we have the first general simple solution of the system (4.5) in the situation that every eigenvalue is complete. In this case we also say that A is diagonalizable.

Theorem 3.3.4. *If the eigenvalues of A are λ_1, λ_2, ..., λ_k for which $de(\lambda_j) = m(\lambda_j)$ for every $j = 1...k$ then the general solution of (4.5) is*

$$x(t) = \sum_{s=1}^{n} c_s e^{\lambda_s t} v_s$$

where $v_1, ..., v_n$ is a basis of eigenvectors.

If we have $de(\lambda) < m(\lambda)$ then we call the eigenvalue λ to be **defective**. Let us notice that if an eigenvalue is not complete it is defective. For a defective eigenvalue the number $m(\lambda) - de(\lambda)$ is called the **defect** of λ.

3.4 Defect one and multiplicity two

Suppose that $Av = \lambda v$ and $(A - \lambda)w = v$. It turns out that in this case there is always a solution in w of this last equation. Let us check that the vector function $u(t) = (w + tv)e^{\lambda t}$ is a solution of (4.5). We have $u'(t) = [\lambda(w + tv) + v]e^{\lambda t}$ and $Au(t) = (\lambda w + v + \lambda tv)e^{\lambda t}$. So, we have $u'(t) = Au(t)$. This solution is linearly independent of $e^{\lambda t}v$. Indeed, if $c_1 e^{\lambda t} v + c_2(w + tv)e^{\lambda t} = 0$ for all t, we can get rid first of $e^{\lambda t}$ to obtain $c_1 v + c_2(w + tv) = 0$ for all t. After differentiation with respect to t we obtain $c_2 v = 0$ which implies $c_2 = 0$ and then automatically $c_1 = 0$. Hence the two vector valued functions that one has to take corresponding to the eigenvalue λ are: $e^{\lambda t} v$ and $e^{\lambda t}(w + tv)$.

Example: Let us take the following example $A := \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 0 & 4 & 1 \end{bmatrix}$ and let us say we want to solve the following initial value problem

$$\begin{cases} x' = Ax, \\ x_1(0) = 1, x_2(0) = 2, x_3(0) = 3. \end{cases}$$

The characteristic polynomial is

$$p(\lambda) = det \begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 0 & 4 & 1-\lambda \end{bmatrix} = det \begin{bmatrix} 0 & (1+\lambda)(3-\lambda)/2 & (1+\lambda) \\ 2 & 1-\lambda & 2 \\ 0 & 4 & 1-\lambda \end{bmatrix} =$$
$$-2(1+\lambda) \begin{bmatrix} (3-\lambda)/2 & 1 \\ 4 & 1-\lambda \end{bmatrix} = -(\lambda+1)^2(\lambda-5).$$

Hence $\lambda_1 = \lambda_2 = -1$ with multiplicity 2 and $\lambda_3 = 5$. Let us solve for the eigenvectors of $\lambda_3 = 5$:

$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ which gives simply a one-dimensional eigenspace $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

We will just take $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Now we solve $(A - \lambda_1)v = 0$: $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ which gives also a one-dimensional eigenspace generated by $v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Then we solve for $(A - \lambda_2)w = v_2$. One solution of this equation can be taken to be $w = \begin{bmatrix} 3/2 \\ 0 \\ -1 \end{bmatrix}$. Therefore the general solution of our equation is $x(t) =$

$c_1v_1e^{5t} + [c_2v_2 + c_3(w + tv_2)]e^{-t}$. This gives

$$\begin{cases} x_1(t) = c_1e^{5t} + [c_2 + c_3(3/2 + t)]e^{-t} \\ x_2(t) = c_1e^{5t} + (c_2 + c_3t)e^{-t} \\ x_3(t) = c_1e^{5t} - [2c_2 + c_3(1 + 2t)]e^{-t} \end{cases}.$$

Then we need to determine the constants c_1, c_2, c_3 in order to get the initial condition satisfied. This gives $c_1 = \frac{19}{9}$, $c_2 = -\frac{1}{9}$, and $c_3 = -\frac{2}{3}$. Finally substituting we get

$$\begin{cases} x_1(t) = \frac{19}{9}e^{5t} - \frac{10+6t}{9}e^{-t} \\ x_2(t) = \frac{19}{9}e^{5t} - \frac{1+6t}{9}e^{-t} \\ x_3(t) = \frac{19}{9}e^{5t} - \frac{8+12t}{9}e^{-t} \end{cases}.$$

3.4.1 Generalized vectors

Let λ be an eigenvalue of A.

Definition 3.4.1. *A vector v is called a* **rank r generalized eigenvector** *associated to λ if $(A - \lambda I)^r v = 0$ and $(A - \lambda I)^{r-1}v \neq 0$.*

Clearly, every eigenvector is a rank one generalized eigenvector. Notice that if v is a rank r generalized eigenvector associated to λ then we can define $v_1 = (A - \lambda I)^{r-1}v$ which is not zero by definition and satisfies $Av_1 = \lambda v_1$. This means v_1 is a regular eigenvector. We define in general $v_2 = (A - \lambda I)^{r-2}v$,..., $v_{r-1} = (A - \lambda)v$, $v_r = v$. These vectors are all not equal to zero because otherwise v_1 becomes zero. In practice we need to work our way backwards in order to determine a generalized eigenvector. First we find v_1 as usual since it is a eigenvector. Then we find v_2 from the equation $(A - \lambda I)v_2 = v_1$. Next we solve for v_3 from the equation $(A - \lambda I)v_3 = v_2$ and so on. One can show that $v_1, v_2, ..., v_r$ are linearly independent. We have the following important theorem from linear algebra which is sometime called the Jordan representation theorem because it allows one to represent the matrix, up to a similarity, i.e. SAS^{-1}, as a direct sum of Jordan blocks: $\lambda I_k + N$ where N is a nilpotent matrix that has ones above the diagonal and zero for the rest of the entries.

Theorem 3.4.2. *For every $n \times n$ matrix A there exists a basis of generalized eigenvectors. For each eigenvalue λ of multiplicity $m(\lambda)$ there exists $m(\lambda)$ generalized linearly independent vectors associated.*

In general, if we have an eigenvector v_1 such that v_r is is a rank r generalized eigenvector corresponding to the eigenvalue λ, the contribution of these to a fundamental set of solutions for (4.5) is with the following set of functions:

$$
\begin{aligned}
&u_1(t) = e^{\lambda t}v_1,\\
&u_2(t) = (v_2 + tv_1)e^{\lambda t},\\
&u_3(t) = (v_3 + tv_2 + \tfrac{t^2}{2!}v_1)e^{\lambda t},\\
&...,\\
&u_r(t) = (v_r + tv_{r-1} + \tfrac{t^2}{2!}v_{r-2} + ... + \tfrac{t^{r-1}}{(r-1)!}v_1)e^{\lambda t}.
\end{aligned}
$$

One can show that these are linearly independent vector-valued functions which are solutions of (4.5). We may have for a certain eigenvalue different sets of this type. Putting all together will give a fundamental set of solutions of (4.5). This fact is insured by the Theorem 3.4.2. In the case the eigenvalue is pure complex we just take the real and imaginary parts of these vector valued functions.

Example: *Find the general solution of the differential system:*

$$
\begin{cases}
x_1' = x_1 + x_2\\
x_2' = x_2 + x_3\\
x_3' = x_3
\end{cases}
$$

The matrix of this system is $A = \begin{bmatrix} 1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1 \end{bmatrix}$. Clearly we have $\lambda = 1$ as eigenvalue of multiplicity 3 and a corresponding eigenvector is $v_1 = \begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}$. Then the equation $(A - \lambda I)v_2 = v_1$ has a nonzero solution $v_2 = \begin{bmatrix} 0\\ 1\\ 0 \end{bmatrix}$. This makes v_2 a rank 2 generalized vector. If we continue, the equation $(A - \lambda I)v_3 = v_2$ has a nontrivial solution $v_3 = \begin{bmatrix} 0\\ 0\\ 1 \end{bmatrix}$. This means that v_3 is a rank 3 generalized vector. Then a fundamental set of solutions of our system is $u_1(t) = e^t v_1$, $u_2 = (tv_1 + v_2)e^t$ and $u_3 = (t^2 v_1/2 + tv_2 + v_3)e^t$. Therefore, the general solution of the differential system is $\begin{cases} x_1 = (c_1 e^t + c_2 t + c_3 t^2/2)e^t\\ x_2 = (c_2 + tc_3)e^t\\ x_3 = c_3 e^t \end{cases}$.

3.4.2 Fundamental Matrix of Solutions, Exponential Matrices

Suppose we are still solving a system $x' = Ax$ as before. A **fundamental matrix of solutions** is a $n \times n$ matrix formed with n linearly independent solutions of the system (4.5). Such a matrix can be simply calculated by using the Taylor formula, $e^x = 1 + x + x^2/2! + ...$, for matrices instead of numbers.

Definition 3.4.3. *The matrix e^{tA} is the result of the infinite series $\sum_{k=0}^{\infty} A^k \frac{t^k}{k!}$, $t \in \mathbb{R}$.*

Theorem 3.4.4. *The matrix e^{tA} is a fundamental matrix of solutions of $x' = Ax$. The solution of the initial value problem $\begin{cases} x' = Ax \\ x(0) = x_0 \end{cases}$ is given by $x(t) = e^{tA}x_0$.*

Example: Let us take Problem 12, page 356 as an exemplification of this. The system given is $\begin{cases} x_1' = 5x_1 - 4x_2 \\ x_2' = 3x_1 - 2x_2 \end{cases}$ with the matrix $A = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$. We need to calculate e^{tA}.

The characteristic polynomial of A is $p(\lambda) = \lambda^2 - 3\lambda + 2$. (In general the characteristic polynomial for 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\lambda^2 - tr(A)\lambda + det(A)$, where $tr(A) = a + d$). The eigenvalues in this case are $\lambda_1 = 1$ and $\lambda_2 = 2$. Two eigenvectors are in this case $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ corresponding to λ_1 and λ_2 respectively. If we denote the matrix $[v_1|v_2]$ by $S := \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$ then let us observe that $AS = [v_1|2v_2] = SD$ where $D := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is a diagonal matrix with entries exactly the two eigenvalues. Therefore $A = SDS^{-1}$. This allows us to calculate $A^n = SD^nS^{-1}$. Notice that $e^{tD} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$. In general the inverse of a 2×2 matrix matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Hence

$$e^{tA} = Se^{tD}S^{-1} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix}$$

or

$$e^{tA} = \begin{bmatrix} -3e^t + 4e^{2t} & 4e^t - 4e^{2t} \\ -3e^t + 3e^{2t} & 4e^t - 3e^{2t} \end{bmatrix}.$$

Homework:

Section 5.4 pages 342-343 Problems 23-33;

Section 5.5 page 356 problems 1-20, 25-30;

Chapter 4

Nonlinear Systems and Qualitative Methods

4.1 Lecture XIII

Quotation: *"I entered an omnibus to go to some place or other. At that moment when I put my foot on the step the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuchsian functions were identical with non-Euclidean geometry." Henri Poincaré*

4.1.1 Nonlinear systems and phenomena, linear and almost linear systems

We are going to discuss the behavior for solutions of autonomous systems DE of the form

$$\begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{cases} \tag{4.1}$$

Let us assume that the two functions F and G are continuous on a region $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 | a < x < b, c < y < d\}$. This region is going to be called **phase plane**. By a similar theorem of existence and uniqueness we have a unique solution to the IVP:

$$\begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \\ x(0) = x_0, y(0) = y_0 \end{cases} \tag{4.2}$$

where $(x_0, y_0) \in \mathcal{R}$. The curve $(x(t), y(t))$, $t \in (-\epsilon, \beta)$ represented in $\mathcal{R}$ is called a **trajectory**. For each point of $\mathcal{R}$ there exist one and only one trajectory containing it.

Definition 4.1.1. *A* **critical point** *for the system (4.5) is a point $(a,b) \in \mathcal{R}$ such that $F(a,b) = G(a,b) = 0$.*

Clearly, if (a,b) is a critical point the constant function $(x(t), y(t)) = (a,b)$ for every $t \in \mathbb{R}$ is a solution of (4.5) which is called an **equilibrium solution**. We observe that the trajectory of a critical point is just a point.

The **phase portrait** is a sketch of the phase plane and a few typical trajectories together with all critical points.

Definition 4.1.2. *A critical point (a,b) is called a* **node** *if either every trajectory approaches (a,b) or every trajectory recedes form (a,b) tangent to a line at (a,b). A node can be a* **sink** *if all trajectories approach the critical point or a* **source** *if all trajectories emanate from it.*

A node can be **proper** or **improper** depending upon the number of tangent lines that the trajectories have: infinitely many or only two. The following notion of stability is the same as for ODE case:

Definition 4.1.3. *A critical point (a,b) is called* **stable** *if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x_0 - a| < \delta$ and $|y_0 - b| < \delta$ the solution of the IVP (4.2) statistics $|x(t) - a| < \epsilon$ and $|y(t) - b| < \epsilon$.*

In general nodal sinks are stable critical points. A critical point which is not stable is called **unstable**.

A critical point can be stable without having the trajectories approach the critical point. If a stable critical point is surrounded by simple closed trajectories representing periodic solutions then such a critical point is called **(stable) center**.

Definition 4.1.4. *A critical point is called* **asymptotically stable** *if it is stable and for some $\delta > 0$ if $|x_0 - a| < \delta$ and $|y_0 - b| < \delta$ then $\lim_{t\to\infty} x(t) = a$ and $\lim_{t\to\infty} y(t) = b$.*

An asymptotically stable critical point with the property that every trajectory spirals around it is called **spiral sink**. A **spiral source** is a critical point as before but with time t going to $-\infty$ instead of ∞. If for a critical point there are two trajectories that approach the critical point but all the other ones are unbounded as $t \to \infty$ then we say the critical point is a **saddle point**. Under certain conditions one can show that there are only four possibilities for trajectories:

1. $(x(t), y(t))$ approaches a critical point (a, b) as $t \to \infty$;
2. $(x(t), y(t))$ is unbounded;
3. $(x(t), y(t))$ is a periodic solution;
4. $(x(t), y(t))$ spirals toward a periodic solution as $t \to \infty$.

4.2 Linear and almost linear systems

In the linear case, as usual, we can always have more to say. A critical point (a, b) is called **isolated** if there are no other critical points in a neighborhood of (a, b). We assume that F and G are differentiable around (a, b). The **linearized sytem** associated to (4.5) is

$$\begin{cases} \frac{dx}{dt} = \frac{\partial F}{\partial x}(a,b)(x-a) + \frac{\partial F}{\partial y}(a,b)(y-b) \\ \frac{dy}{dt} = \frac{\partial G}{\partial x}(a,b)(x-a) + \frac{\partial G}{\partial y}(a,b)(y-b). \end{cases} \tag{4.3}$$

A system is said to be **almost linear** at the isolated critical point (a, b) if

$$\lim_{x\to a, y\to b} \frac{F(x,y) - \frac{\partial F}{\partial x}(a,b)(x-a) - \frac{\partial F}{\partial y}(a,b)(y-b)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$

and

$$\lim_{x\to a, y\to b} \frac{G(x,y) - \frac{\partial G}{\partial x}(a,b)(x-a) - \frac{\partial G}{\partial y}(a,b)(y-b)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

In this case its linearization is (4.8). This is a linear system with constant coefficients. In practice, the functions F and G are continuously differentiable. This insures that the system (4.5) is almost linear around a critical point.

Eigenvalues λ_1,λ_2		Type of critical point
Real, different, same sign		Improper node
Real, unequal, opposite signs		Saddle
Real and equal,		Proper or improper node
Pure complex,		Spiral point
Pure imaginary		Center

Table 4.1: Type of critical points for linear systems

4.3 Critical points classification of linear systems

We are going to consider only two dimensional linear systems of the type

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \tag{4.4}$$

which we already know how to solve. Let us assume that the two eigenvalues are λ_1 and λ_2. Then, what type of critical point $(0,0)$ of (4.4) is could be determined by the following chart:

For the stability of the critical point of (4.4) we have:

Theorem 4.3.1. *Let λ_1 and λ_2 be the eigenvalues of the system (4.4) which has $(0,0)$ as an isolated critical point. The critical point $(0,0)$ is*

(a) Asymptotically stable if the real parts of λ_1 and λ_2 are both negative;

(b) Stable but not asymptotically stable if the real parts of λ_1 and λ_2 are both zero;

(c) Unstable if either λ_1 or λ_2 has a positive real part.

The next theorem says that for an almost liner system the effect of small perturbations around an isolated critical point is almost determined by its linearization at that point.

Theorem 4.3.2. *Let λ_1 and λ_2 be the eigenvalues of the linearization system at an isolated critical point of an almost linear system (4.5). Then*

(a) If $\lambda_1 = \lambda_2$, we have a node or spiral point; in this case the critical point is asymptotically stable if $\lambda_1 = \lambda_2 < 0$ or unstable if $\lambda_1 = \lambda_2 > 0$.

(b) If λ_1 and λ_2 are pure imaginary, then we have either a center or a spiral point (and undetermined stability)

(c) Otherwise the type and stability of (a, b) is the same as the one for the linearization system.

Homework:

Section 6.1 pages 375-376 Problems 1-8;

Section 6.2 page 389 Problems 1-32;

4.4 Lecture XIV

> **Quotation:** *"Everything is vague to a degree you do not realize till you have tried to make it precise." Bertrand Russell British author, mathematician and philosopher (1872 - 1970)*

4.4.1 Nonlinear spring

Let us assume that Hooks's law is "corrected" a little to: $F = -kx + \beta x^3$. In a sense, it is natural to think that there are some other terms in the Taylor expansion of the force acting on a body of mass m in terms of the displacement x. In this case the differential equation that we obtain from Newton's law is $mx'' = -kx + \beta x^3$. We can turn this into a system if we introduce $y = \frac{dx}{dt}$:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{k}{m}x + \frac{\beta}{m}x^3. \end{cases} \tag{4.5}$$

The discussion of what happens with the mechanical system depends clearly on β.

Case $\beta < 0$ "hard spring": In this situation we have only one critical point: $(0,0)$. The Jacobian of the almost linear system (4.5) is

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} + 3\frac{\beta}{m}x^2 & 0 \end{bmatrix}$$

so at the critical point $J(0,0) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}$

The eigenvalues of $J(0,0)$ are $\pm i\sqrt{\frac{k}{m}}$ and according to the theorem about transfer of stability and type from the previous lecture we see that all we can say is that $(0,0)$ is either a center or a spiral point. From the point stability the theorem does not say what happens.

But we can determine what is going on by integrating (4.5). By multiplying the second equation in (4.5) by y and integrating we obtain

$$\frac{y^2}{2} + \frac{kx^2}{2m} + \frac{|\beta|x^4}{4m} = constant.$$

These curves are almost like circles around $(0,0)$ (see figure below). The stability but not asymptotic stability of $(0,0)$ follows.

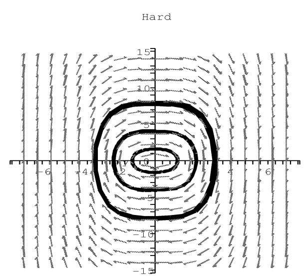


Figure 1

This means that we have almost regular oscillations around the equilibrium position.

Case $\beta > 0$ "soft spring": The system has two more critical points: $(\pm\sqrt{\frac{k}{\beta}}, 0)$. The Jacobian at the critical points is at the critical point $J(\pm\sqrt{\frac{k}{\beta}}, 0) = \begin{bmatrix} 0 & 1 \\ \frac{2k}{m} & 0 \end{bmatrix}$ which means the eigenvalues are real one positive and on negative:$\pm\sqrt{\frac{2k}{m}}$. so by the same theorem we have that both the new critical points are saddle points and unstable.

In fact if we use the method of integration as before we get

$$\frac{y^2}{2} + \frac{kx^2}{2m} - \frac{\beta x^4}{4m} = c.$$

The phase portrait is as below:

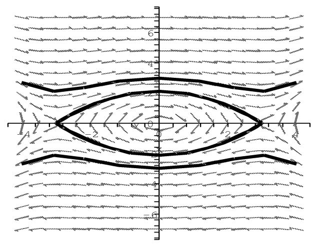


Figure 2

This suggests that for a certain constant c (say c_0, the Energy) we get solutions which go directly to an static position without any oscillations. For values of c smaller than that quantity we have oscillations around the equilibrium position. For values of the energy bigger than c_0 the displacement goes to infinity. This is not corresponding to anything real so it seems like this model does not work.

4.4.2 Chaos

Let us start with the logistic differential equation

$$\frac{dP}{dt} = aP - bP^2, \quad (a, b > 0) \tag{4.6}$$

which models in general a bounded population.

We are going to look into the discretization of this equation according to Euler's method: compute $P_{n+1} - P_n = (aP_n - bP_n^2)h$ which are the values of the approximation of $P(t)$ at the times $t_k = t_0 + kh$, where h is the step size. This can written as $P_{n+1} = rP_n - sP_n^2$ where $r = 1 + ah$ and $s = bh$. Then if we substitute $P_n = \frac{r}{s}x_n$ then the recurrence becomes

$$x_{n+1} = rx_n(1 - x_n). \tag{4.7}$$

We notice that the function $f(x) = rx(1-x)$ has a maximum at $x = 1/2$ equal to $r/4$. So, for $r < 4$ this function maps the interval $[0, 1]$ into itself. So one can iterate and compute the sequence defined in (4.7) for large values of n. It turns out that for $r < r_c$ where $r_c \approx 3.57..$ the behavior of the sequence can be predicted in the sense that approaches a certain number of limit points. The sequence becomes chaotic for $r = r_c$, i.e. there are infinitely many limit points that fill out the interval $[0, 1]$. In other words the sequence is unpredictable. For instance we cannot tell, unless we compute it precisely where $x_{1000000}$ is in the interval $[0, 1]$.

The number of periodic orbits for $r < r_c$ changes at some specific values r_k. The mathematician Mitchell Feigenbaum discovered that

$$\lim_{n\to\infty} \frac{r_k - r_{k-1}}{r_{k+1} - r_k} = constant \ \approx \ 4.669...$$

This constant appears in other places in mathematics, so it gained a status similar to those constants such as π, e or γ. Some differential equations have the same type of behavior for certain values of the parameters. Such examples are $mx'' + cx' + kx + \beta x^3 = F_0 \cos \omega t$ (forced Duffing equation) or the famous Lorenz system

$$\begin{cases} \frac{dx}{dt} = -sx + sy \\ \frac{dy}{dt} = -xz + rx - y \\ \frac{dz}{dt} = xy - bz. \end{cases} \tag{4.8}$$

Homework:

Section 6.4 page 418, Problems 2, 5-8;

Section 7.1 pages 444-445, Problems 1-42.

Chapter 5

Laplace Transform

5.1 Lecture XV

Quotation: *"Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective positions of the beings which compose it, if moreover this intelligence were vast enough to submit these data to analysis, it would embrace in the same formula both the movements of the largest bodies in the universe and those of the lightest atom; to it nothing would be uncertain, and the future as the past would be present to its eyes." Pierre Simon De Laplace* $(1749-1827)$*, French mathematician, philosopher. Theorie Analytique de Probabilites: Introduction, v. VII, Oeuvres (1812-1820).*

5.1.1 Definition and a few examples

The Laplace transform is a transformation on functions as the operator D of differentiation that we have encountered earlier. The study of it in this course is motivated by the fact that some differential equations can be converted via the Laplace transform into an algebraic equation. This is in general thought as being easier to solve and then one obtains the solution of the given differential equation by taking the inverse Laplace transform for the solution of the the corresponding algebraic equation.

In order to define this transform we need a few definitions beforehand.

Definition 5.1.1. *A function f is called piecewise continuous on the interval $[a,b]$ if there is a partition of the interval $x_0 = a < x_1 < x_2 < ... < x_n = b$ such that f is* *continuous on each interval* (x_k, x_{k+1}) *and it has sided limits at each point* x_k.

A function f defined on an unbounded interval is said to be piecewise continuous if it is so on each bounded subinterval.

Definition 5.1.2. *A function* $f : [0,\infty) \to \mathbb{R}$ *is called of exponential type at* ∞ *if there exist nonnegative constants* M, c *and* T *such that* $|f(t)| \le Me^{ct}$ *for all* $t \ge T$.

Definition 5.1.3. *For every function* $f : [0,\infty) \to \mathbb{R}$ *which is piecewise continuous on some interval* $[T,\infty)$, *integrable on* $[0,T]$ *and of exponential type at infinity the Laplace transform* $\mathcal{L}(f)$ *is the new function of the variable* s *defined by*

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)dt.$$

The domain of $\mathcal{L}(f)$ *is taken to be the set of all* s *for which the improper integral exists, i.e.* $\lim\limits_{n\to\infty} \displaystyle\int_0^n e^{-st} f(t)dt$ *exists.*

The next theorem tells us that the above definition is meaningful. We are going to denote the class of these functions by $\mathcal{D}(\mathcal{L})$.

Theorem 5.1.4. *Under the assumption in the definition (5.1.3)the Laplace transform* $\mathcal{L}(f)(s)$ *exists for every* $s > c$.

Before we prove this theorem let us compute the Laplace transform for some simple functions.

Example 1: Suppose we take $f(t) = 1$ for all $t \in [0,\infty)$. Then $\mathcal{L}(f)(s) = \int_0^\infty e^{-st}dt = -\frac{e^{-st}}{s}|_0^\infty = \frac{1}{s}$, for all $s > 0$. Therefore we write

$$\mathcal{L}(1)(s) = \frac{1}{s},\ s > 0.$$

Example 2: Let us take $f(t) = e^{ct}$ for all $t \ge 0$. Then if $s > c$, $\mathcal{L}(f)(s) = \int_0^\infty e^{-st}e^{ct}dt = \int_0^\infty e^{-(s-c)t}dt = -\frac{e^{-(s-c)t}}{s-c}|_0^\infty = \frac{1}{s-c}$. Hence

$$\mathcal{L}(e^{c\cdot})(s) = \frac{1}{s-c},\ s > c.$$

Proof of Theorem 5.1.4. We need to show that the limit $\lim\limits_{n\to\infty} \displaystyle\int_0^n e^{-st} f(t)dt$ exists. Using Cauchy's characterization of the existence of a limit it suffices to show that $\lim\limits_{n,m\to\infty} \displaystyle\int_m^n e^{-st} f(t)dt = 0$. We have the estimate

$$
\begin{aligned}
& |\int_m^n e^{-st} f(t)dt| \leq \int_m^n e^{-st}|f(t)|dt \leq M \int_m^n e^{-(s-c)t}dt = \\
(5.1) \qquad & \tfrac{M}{(s-c)}(e^{-(s-c)m} - e^{-(s-c)n}) \to 0
\end{aligned}
$$

as $m, n \to \infty$ $(T \leq m < n)$ provided that $s > c$. ∎

Note: Cauchy's characterization is one of the most common tools in analysis. Augustin Louis Cauchy was born on 21st of August, 1789 in Paris, France, and died May 23rd, 1857 in Sceaux near Paris.

5.1.2 General Properties

Corollary 5.1.5. *Let us assume that f is as in the Theorem 5.1.4. Then* $\lim_{s\to\infty} \mathcal{L}(f)(s) = 0$.

Proof. Since we know the limit in the definition of $\mathcal{L}(f)(s)$ exists we let n go to infinity in the sequence of inequalities (5.1) but fix $m = T$. That gives

$$|\mathcal{L}(f)(s)| \leq |\int_0^m e^{-st} f(t)dt| + \frac{M}{s-c},$$

for every $s > c$ and the conclusion of our corollary follows from this and a theorem of convergence under the integral sign. ∎

The Laplace transform may exist even for functions that are unbounded on a finite interval. One such example is $f(t) = t^a$, $t > 0$ with $a > -1$. Notice that for $a \in (-1, 0)$ the integral $\int_0^\infty e^{-st} t^a dt$ is also improper at 0. To compute $\mathcal{L}(f)(s)$ we change the variable $st = u$ $(s > 0)$ and obtain

$$\mathcal{L}(f)(s) = \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a du = \frac{\Gamma(a+1)}{s^{a+1}},$$

where Γ is defined by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dx$ and exists for all $x > 0$. An integration by parts shows that

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dx = -e^{-t} t^x|_{t=0}^{t=\infty} + x \int_0^\infty e^{-t} t^{x-1} dx = x\Gamma(x).$$

because $\Gamma(1) = 1$ we get by induction $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

In particular we get for instance $\mathcal{L}(t^5)(s) = \frac{\Gamma(6)}{s^6} = \frac{5!}{s^6}$, $s > 0$. For fractional values of a one needs to know $\Gamma(a)$. One interesting fact here is that $\Gamma(1/2) = \sqrt{\pi}$. To see this let us change the variable $t = u^2$ in $\Gamma(1/2) = \int_0^\infty e^{-t} t^{-1/2} dt$.

We obtain $\Gamma(1/2) = \int_0^\infty e^{-u^2}u^{-1}2udu = 2\int_0^\infty e^{-u^2}du$. Now we calculate $\Gamma(1/2)^2 = 4\int_0^\infty e^{-u^2}du\int_0^\infty e^{-v^2}dv = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}dudv$. We then use polar coordinates $u = r\cos s$, $v = r\sin s$. So we get $\Gamma(1/2)^2 = 4\int_0^{\pi/2}(\int_0^\infty e^{-r^2}rdr)ds = \pi$ which implies $\Gamma(1/2) = \sqrt{\pi}$. This allows one to compute for instance $\Gamma(3/2) = \frac{3}{2}\Gamma(1/2) = \frac{3\sqrt{\pi}}{2}$.

Proposition 5.1.6. *The Laplace transform is linear.*

Proof. The integral and the limit are linear transformations on functions. One needs to check also that $\mathcal{D}(\mathcal{L})$ is a linear space of functions. ■

Example: $\mathcal{L}(3t^2+2\sqrt{t})(s) = 3\mathcal{L}(t^2)(s)+2\mathcal{L}(\sqrt{t})(s) = 3\dfrac{2!}{s^3}+2\dfrac{\Gamma(3/2)}{s^{3/2}} = \dfrac{6}{s^3}+\dfrac{3\sqrt{\pi}}{s\sqrt{s}}$.

Another example we would like to do involves the Laplace transform of a complex valued function which is a natural extension of the Laplace transform of real valued functions.

Example: We take $f(t) = e^{zt}$ where $z = a+ib$. Notice that $|f(t)| = e^{at}$ so this function is of exponential type at infinity. We have $\mathcal{L}(f)(s) = \displaystyle\int_0^\infty e^{-st}e^{zt}dt = \int_0^\infty e^{-(s-z)t}dt = \lim_{t\to\infty}\left(\frac{1}{s-z}-\frac{e^{-(s-z)t}}{s-z}\right) = \frac{1}{s-z}$, provided that $s > a$.

This is happening because $|\frac{e^{-(s-z)t}}{s-z}| = \frac{e^{-(s-a)t}}{|s-z|} \to 0$ as $t \to \infty$. Since $\mathcal{L}$ is linear $Re\mathcal{L}(f)(s) = \mathcal{L}(e^{at}\cos bt)(s) = Re\frac{1}{s-z} = \frac{s-a}{(s-a)^2+b^2}$ and $Im\mathcal{L}(f)(s) = \mathcal{L}(e^{at}\sin bt)(s) = Im\frac{1}{s-z} = \frac{b}{(s-a)^2+b^2}$.

Example: This example involves the Laplace transform of a function denoted by u and defined by

$$u(t) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0, \end{cases}$$

or a translation of u which is denoted by u_a and defined as $u_a(t) = u(t-a)$, $t \in \mathbb{R}$.

We get $\mathcal{L}(u_a)(s) = \int_0^\infty e^{-st}u_a(t)dt = \int_a^\infty e^{-st}dt = \frac{e^{-sa}}{s}$. Let us record the main formulas that we have discovered so far in the table 5.1.2:

Proposition 5.1.7. *The Laplace transform is a one-to-one map in the following sense: $\mathcal{L}(f)(s) = \mathcal{L}(g)(s)$ for all $s > s_0$ implies that the functions f and g coincide at all their continuity points.*

Since this is true we are allowed to take the inverse of the Laplace transform denoted by $\mathcal{L}^{-1}$ by simply inverting the table above (in other words it is not ambiguous to talk about the inverse of the Laplace transform).

$f(t)$	$\mathcal{L}(f)(s)$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$u_a(t)$	$\frac{e^{-sa}}{s}$
t^a $(a>-1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$
e^{at}	$\frac{1}{s-a}$

Table 5.1: Laplace Transform Formulae

Theorem 5.1.8. *Given the function $f:[0,\infty)\to\mathbb{R}$ of exponential type at infinity which is continuous and whose derivative is piecewise continuous, then $\mathcal{L}(f')$ exists and $\mathcal{L}(f')(s)=s\mathcal{L}(f)(s)-f(0)$.*

Proof. First we assume that the derivative is continuous at all points. Then an integration by parts will give

$\mathcal{L}(f)(s)=\int_0^\infty e^{-st}f'(t)dt=e^{-st}f(t)|_0^\infty+s\int_0^\infty e^{-st}f(t)dt=s\mathcal{L}(f)(s)-f(0)$, where $\lim_{t\to\infty}e^{-st}f(t)=0$ because of the hypothesis on f to be of exponential type.

The proof in the general case goes the same way with the only change that the fundamental formula of calculus holds true for f under the given hypothesis. ■

Corollary 5.1.9. *If the function f is of exponential type and it has derivatives of order k, ($k\le n$), exist with $f^{(n)}$ piecewise continuous then*

$$\mathcal{L}(f^n)(s)=s^n\mathcal{L}(f)(s)-s^{n-1}f(0)-...-f^{(n-1)}(0).$$

Let us solve a differential equation using the Laplace transform now. Problem 6, page 455 asks for the following initial value problem of a second order linear DE with constant coefficients but non-homogeneous: $x''+4x=\cos t$, $x(0)=x'(0)=0$. We first apply the Laplace transform to both sides of the equation and use the formula for the Laplace transform of the derivative of a function: $s^2\mathcal{L}(x)(s)-sx(0)-x'(0)+4\mathcal{L}(x)=\frac{s}{s^2+1}$. Hence we get $\mathcal{L}(x)(s)(s^2+4)=\frac{s}{s^2+1}$. Solving for $\mathcal{L}(x)(s)$ we obtain

$$\mathcal{L}(x)(s)=\frac{s}{(s^2+1)(s^2+4)}. \tag{5.2}$$

In order to take the inverse Laplace transform we need to write the right hand side of (5.2) in its partial fraction decomposition. There are some shortcuts that one can use in order to obtain the partial fraction decomposition. These techniques will be discussed in class. In this simple case it is easy to see that we have

$$\mathcal{L}(x)(s) = \frac{1}{3}\frac{s}{(s^2+1)} - \frac{1}{3}\frac{s}{(s^2+4)}.$$

Equivalently, if we remember the table 5.1.2 of Laplace transforms we can rewrite this equality as

$$\mathcal{L}(x)(s) = \frac{1}{3}\mathcal{L}(\cos t)(s) - \frac{1}{3}\mathcal{L}(\cos 2t).$$

Because the Laplace transform is linear and injective we conclude that $x(t) = \frac{1}{3}\cos t - \frac{1}{3}\cos 2t$ for all t.

Theorem 5.1.10. *If f is piecewise continuous and of exponential type at infinity then*

$$\mathcal{L}(\int_0^t f(x)dx)(s) = \frac{1}{s}\mathcal{L}(f)(s).$$

Proof. One can see that $g(t) = \int_0^t f(x)dx$ is continuous and whose derivative is piecewise continuous. It is easy to see that it is also of exponential type. Hence one can apply the Theorem 5.1.8 to g: $\mathcal{L}(g')(s) = s\mathcal{L}(g)(s) - g(0)$. This is exactly the identity that we want to establish. ■

Homework:

Section 7.2 page 455, Problems 1-37.

5.2 Lecture XVI

Quotation: *"Mathematics is the search of structure out there in the most incredible places of the human intellect and at the same time apparently unrelated, in all the corners of what presents itself as reality." Anonymous*

5.3 More properties of the Laplace transform

We have shown how to obtain the Laplace transform for the functions in the table below:

$f(t)$	$\mathcal{L}(f)(s)$
t^n, $n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$, $s > 0$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$u_a(t)$	$\frac{e^{-sa}}{s}$, $s > 0$
t^a $(a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$, $s > 0$
e^{zt}	$\frac{1}{s-z}$, $s > Rez$

Using the theorem about the Laplace transform of the derivative of a function we may obtain additional transforms using the technique exemplified in the next example:

Example: Problem 27, page 456. We consider the function $f_n(t) = t^n e^{zt}$ with $z = a + ib$ and $n \in \mathbb{N}$. Then f is continuous of exponential type ($c = Re\ z$) and its derivative exists everywhere, $f_n'(t) = nt^{n-1}e^{zt} + zt^n e^{zt}$, and f' is piecewise continuous (in fact continuous on $[0, \infty)$). Hence $\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$ or $n\mathcal{L}(f_{n-1})(s) + z\mathcal{L}(f_n)(s) = s\mathcal{L}(f_n)(s)$. Solving for $\mathcal{L}(f_n)(s)$ we get

$$\mathcal{L}(f_n)(s) = \frac{n}{s-z}\mathcal{L}(f_{n-1})(s),\ s > Re\ z.$$

This recurrence can be used inductively to prove then that

$$\mathcal{L}(t^n e^{zt})(s) = \frac{n!}{(s-z)^{n+1}},\ s > Re\ z. \tag{5.3}$$

Let us observe that this formula generalizes several of the formulas that we have seen so far but will give two new ones if we take the real part and the imaginary part of both sides ($z = a + ib$):

$$\mathcal{L}(t^n e^{at}\cos bt)(s) = \frac{n! \sum_{0 \le j \le (n+1)/2} \binom{n+1}{2j}(-1)^j (s-a)^{n+1-2j} b^{2j}}{[(s-a)^2 + b^2]^{n+1}}, \tag{5.4}$$

and

$$\mathcal{L}(t^n e^{at}\sin bt)(s) = \frac{n! \sum\limits_{0\le j\le n/2} \binom{n+1}{2j+1}(-1)^j (s-a)^{n-2j} b^{2j+1}}{[(s-a)^2+b^2]^{n+1}}. \tag{5.5}$$

Using the theorem about the Laplace transform of the integral of a function we may also obtain some inverse Laplace transforms of functions that contain a power of s at the denominator. We again use an example to exemplify this.

Example: Problem 24, page 456. In this problem we need to find the Laplace transform of $F(s) = \frac{1}{s(s+1)(s+2)}$. Because $\mathcal{L}(\int_0^t f(x)dx)(s) = \frac{\mathcal{L}(f)(s)}{s}$ we see that if the right hand side is $F(s)$ we need to find what the inverse Laplace transform just for

$$G(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} = \mathcal{L}(e^{-t})(s) - \mathcal{L}(e^{-2t})(s).$$

Therefore $\mathcal{L}^{-1}(G)(t) = e^{-t} - e^{-2t}$ and our function is $f(t) = \displaystyle\int_0^t e^{-x} - e^{-2x}dx = 1 - e^{-x} - (\frac{1}{2} - \frac{1}{2}e^{-2t}) = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}$.

Next we are going to generalize the theorem about the Laplace transform of the derivative of a function.

Theorem 5.3.1. *Suppose $f : [0,\infty) \to \mathbb{C}$ is piecewise continuous of exponential type (of constant c), which has a derivative f' at the points of continuity with the exception of maybe an isolated set of points. Then*

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0) - \sum_{\substack{t \text{ discontinuity} \\ \text{point of } f}} e^{-st}[f(t+0) - f(t-0)], \quad s > c. \tag{5.6}$$

Proof. Let us assume that the origin $t=0$ and the discontinuity points of f are $\{t_n\}_{n\in\mathcal{D}}$; so $t_1 = 0 < t_2 < ...$, $\mathcal{D} \subset \mathbb{N}$. Next we assume first that f' exists on each interval (t_k, t_{k+1}). Then $\mathcal{L}(f')(s) = \lim\limits_{a\to\infty}\displaystyle\int_0^a e^{-st}f'(t)dt = \lim\limits_{a\to\infty}[\sum_{n=1}^{n(a)+1}\int_{t'_n}^{t'_{n+1}} e^{-st}f'(t)dt]$ where $n(a)$ is the greatest index for which $t_{n(a)} < a$ and $t'_k = t_k$ if $k \le n(a)$ and $t'_{n(a)+1} = a$.

On each interval $[t'_n, t'_{n+1}]$ we apply the integration by parts to the function which becomes continuous at the endpoints when f is extended with the sided limits:

$\int_{t'_n}^{t'_{n+1}} e^{-st} f'(t)dt = e^{-st} f(t)|_{t=t'_n+0}^{t=t'_{n+1}-0} + s\int_{t'_n}^{t'_{n+1}} e^{-st} f(t)dt$. Then

$$\mathcal{L}(f')(s) = \lim_{a\to\infty} s\int_0^a e^{-st} f(t)dt + \lim_{a\to\infty} [\sum_{n=1}^{n(a)+1} e^{-st_{n+1}} f(t'_{n+1} - 0) - e^{-st_n} f(t'_n + 0)]$$

. Rearranging the summation and letting $a \to \infty$ we obtain (5.6). The general case is handled the same way with the observation we made before that the fundamental formula of calculus works under our more relaxed assumptions. ■

For an application let us work Problem 34, page 456. We apply formula (5.6) for $f(x) = (-1)^{\lfloor x \rfloor}$ where $\lfloor x \rfloor$ is the greatest integer function. Figure 1 below gives an idea of what the graph of f looks like.

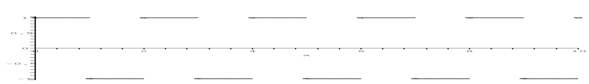


Figure 1

This function has a discontinuity for every $n \in \mathbb{N}$ and a jump of $f(2n+0) - f(2n-0) = 1-(-1) = 2$ for even discontinuity points and $f(2n+1+0) - f(2n+1-0) = -1-1 = -2$ for every odd one. In other words $f(n+0) - f(n-0) = 2(-1)^n$ for every $n \in \mathbb{N}$. Since the derivative is basically zero where it exists applying (5.6) we obtain $0 = s\mathcal{L}(f)(s) - f(0) - \sum_{n=1}^{\infty} 2(-1)^n e^{-ns}$. Using the formula for summing a sequence in geometric progression ($1 + r + r^2 + ... = \frac{1}{1-r}$, whenever $r < 1$) this turns into

$$s\mathcal{L}(f)(s) = 1 + 2(-e^{-s})\frac{1}{1+e^{-s}} = \frac{1-e^{-s}}{1+e^{-s}}.$$

Another way of writing this using the hyperbolic functions $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$ is $\mathcal{L}(f)(s) = \dfrac{1}{s}\dfrac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}} = \dfrac{1}{s}\tanh(s/2)$.

Theorem 5.3.2. (Translation along the s-axis) *If f is such that $\mathcal{L}(f)(s)$ exists for all $s > c$ then $\mathcal{L}(e^{at}f(t))(s)$ exists for all $s > a + c$ and*

$$\mathcal{L}(e^{at}f(t))(s) = \mathcal{L}(f)(s - a).$$

The proof of this theorem is straightforward. Let us work out a few other examples from this next homework:

Problem 10, page 465. *Find the inverse Laplace transform of the function* $F(s) = \dfrac{2s-3}{9s^2-12s+20}$.

Observe that

$$F(s) = \frac{2s-3}{(3s-2)^2+16} = \frac{2s-3}{9[(s-2/3)^2+16/9]}, \text{ or}$$

$$F(s) = \frac{2}{9}\frac{s-2/3}{(s-2/3)^2+(4/3)^2} - \frac{5}{36}\frac{4/3}{(s-2/3)^2+(4/3)^2}.$$

Hence $\mathcal{L}^{-1}(F)(t) = \frac{2}{9}e^{2t/3}\cos 4t/3 - \frac{5}{36}e^{2t/3}\sin 4t/3$.

Problem 34, page 465. The DE is $x^{(4)} + 13x'' + 36x = 0$ with initial conditions $x(0) = x''(0) = 0$, $x'(0) = 2$ and $x'''(0) = -13$. Applying the Laplace transform we get $s^4\mathcal{L}(x) - s^3x(0) - s^2x'(0) - sx''(0) - x'''(0) + 13[s^2\mathcal{L}(x) - sx(0) - x'(0)] + 36\mathcal{L}(x) = 0$.

Substituting the initial conditions we get

$$\mathcal{L}(x)(s)(s^4 + 13s^2 + 36) - 2s^2 + 13 - 26 = 0.$$

Solving for $\mathcal{L}(x)$ gives $\mathcal{L}(x)(s) = \dfrac{2s^2+13}{s^4+13s^2+36} = \dfrac{s^2+4+s^2+9}{(s^2+4)(s^2+9)}$, or $\mathcal{L}(x)(s) = \dfrac{1}{2}\dfrac{2}{s^2+4} + \dfrac{1}{3}\dfrac{3}{s^2+9} = \dfrac{1}{2}\mathcal{L}(\sin 2t)(s) + \dfrac{1}{3}\mathcal{L}(\sin 3t)(s)$.

Taking the inverse Laplace transform we obtain $x(t) = \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t$.

Problem 24, page 465 *Find the inverse Laplace transform for the function* $F(s) = \dfrac{s}{s^4+4a^4}$. Using the idea given in the textbook we factor the denominator of fraction in F:

$$F(s) = \frac{s}{(s^2-2as+2a^2)(s^2+2as+2a^2)} = \frac{s}{[(s-a)^2+a^2][(s+a)^2+a^2]}$$

$$= \frac{1}{4a}\left(\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right) = \frac{1}{4a^2}\left(\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right).$$

Hence $\mathcal{L}^{-1}(F)(t) = \dfrac{1}{4a^2}\left(e^{at}\sin at - e^{-at}\sin at\right)$ or

$$\mathcal{L}^{-1}(F)(t) = \frac{1}{2a^2}\sinh at \sin at.$$

Homework:

Section 7.3 page 455, Problems 1-38.

5.4 Lecture XVII

> **Quotation:** *"The key to make progress in the process of learning mathematics is to ask the right questions." Anonymous*

5.4.1 Convolution of two functions

Let us assume we have two functions, f and g, which are piecewise continuous and of exponential type (with the same constant c)

Definition 5.4.1. *The convolution of f and g is the new function $(f \star g)(t) = \int_0^t f(x)g(t-x)dx$, $t \geq 0$.*

The convolution defined this way is *commutative*: $f \star g = g \star f$. This can be easily seen by a change of variables: $y = t - x$,

$$\begin{aligned}&(f \star g)(t) = \int_0^t f(x)g(t-x)dx = \int_t^0 f(t-y)g(y)(-dy)\\ &= \int_0^t g(y)f(t-y)dy = (g \star f)(t).\end{aligned}$$

Theorem 5.4.2. *The convolution of the two functions of exponential type (with constant c) is also of exponential type (with constant $c+\epsilon$). The Laplace transform of the convolution of two functions is the product of the individual Laplace transforms:*

$$\mathcal{L}(f \star g)(s) = \mathcal{L}(f)(s)\mathcal{L}(g)(s), s > c$$

Proof. Let us denote by $F(s)$ the Laplace transform of f and by $G(s)$ the Laplace transfrom of g. If $s > c$ then $F(s)G(s) = \int_0^\infty e^{-st}f(t)dt \int_0^\infty e^{-sx}g(x)dx$. The function of two variables $(t, x)- > e^{-st}f(t)e^{-sx}g(x)$ is absolutely integrable over the domain $[0,\infty)\times[0,\infty)$ with the same proof as we did when we showed the existence of the Laplace transform. Then we can rewrite $F(s)G(s) = \int_0^\infty \int_0^\infty e^{-s(t+x)}f(t)g(x)dtdx$. We can make a substitution now $t = u$ and $t + x = v$. The domain $[0,\infty) \times [0,\infty)$ can now be described as $\{(v, u) : v \in [0,\infty) and\ u \in [0, v]\}$. The Jacobian of the transformation is $J(x, t) = det \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = 1$, so, the double integral becomes

$$F(s)G(s) = \int_0^\infty e^{-sv}\left[\int_0^v f(u)g(v-u)du\right]dv = \mathcal{L}(f * g)(s).$$

The double integral can be understood in a limit sense (over rectangles) which makes the above computation possible. ■

Let us use this theorem to solve Problem 14, page 474. We need to find the inverse Laplace transform of the function $F(s) = \frac{s}{s^4+5s^2+4}$. We can rewrite $F(s) = \frac{s}{(s^2+1)(s^2+4)} = \mathcal{L}(\sin t)(s)\mathcal{L}(\cos 2t)(s)$

So, $\mathcal{L}^{-1}(F)(t) = \int_0^t \sin x \cos 2(t-x) = \int_0^t \frac{1}{2}[\sin(2t-x) + \sin(3x-2t)]dx$. Then

$$\mathcal{L}^{-1}(F)(t) = \frac{1}{2}\cos(2t-x)|_{x=0}^{x=t} - \frac{1}{6}\cos(3x-2t)|_{x=0}^{x=t} =$$

$$\frac{1}{2}(\cos t - \cos 2t) - \frac{1}{6}(\cos t - \cos 2t) = \frac{1}{3}(\cos t - \cos 2t), t \geq 0.$$

Theorem 5.4.3. [Integration of Transform formula] *Suppose that f is piecewise continuous for $t \geq 0$, has exponential type at infinity (with constant c) and that $\lim\limits_{t\to 0^+} \frac{f(t)}{t}$ exists . If the Laplace transform of f is F, then the improper integral $\int_s^\infty F(u)du$ exists for every $s > c$ and*

$$\mathcal{L}\left(\frac{f(t)}{t}\right)(s) = \int_s^\infty F(u)du, s > c.$$

Proof. Since $F(s) = \int_0^\infty e^{-st}f(t)dt$ the function F is continuous. The improper integral $\int_s^\infty F(u)du$ exists because of the estimate we got when we proved the existence of the Laplace transform. Then

$$\int_s^\infty F(u)du = \int_s^\infty \int_0^\infty e^{-ut}f(t)dtdu.$$

It turns out that the function of two variables $e^{-ut}f(t)$ is integrable on the domain $[s,\infty)\times[0,\infty)$ in the sense of limits on arbitrary rectangles and so the interchange of the integrals is possible. Thus

$$\int_s^\infty F(u)du = \int_0^\infty \left[\int_s^\infty e^{-ut}du\right] f(t)dt = \int_0^\infty e^{-st}\frac{f(t)}{t}dt.$$

■

As we can see the hypothesis that $\lim\limits_{t\to 0^+} \frac{f(t)}{t}$ exists can be relaxed to the existence of the integral $\int_0^1 \frac{|f(t)|}{t}dt$.

We are going to work out Problem 20, page 474. We need to find the Laplace transform of $g(t) = \frac{1-\cos 2t}{t}$. Consider the map $f(t) = 1 - \cos 2t$. We have $\mathcal{L}(f)(s) =$

$\frac{1}{s}-\frac{s}{s^2+4}$. Since $\lim_{t\to 0^+}\dfrac{f(t)}{t}=0$ we can apply Theorem 5.4.3 and obtain $\mathcal{L}(g)(s)=\displaystyle\int_s^\infty \frac{1}{u}-\frac{u}{u^2+4}du=\ln\frac{u}{\sqrt{u^2+4}}|_s^\infty=\ln\frac{\sqrt{s^2+4}}{s}$ for all $s>0$.

Theorem 5.4.4. [Differentiation of the transform] *If f is piecewise continuous and of exponential type (with constant c), then if F is the Laplace transform of f we have*

$$\mathcal{L}(-tf(t))(s)=F'(s),\quad s>c.$$

Proof. Since $F(s)=\int_0^\infty e^{-st}f(t)dt$ and $\mathcal{L}(-tf(t))(s)=\int_0^\infty e^{-st}(-tf(t))dt$ exist, we can calculate

$$\frac{F(s)-F(s_0)}{s-s_0}-\mathcal{L}(-tf(t))(s_0)=\int_0^\infty\frac{e^{-st}-e^{-s_0t}-(s-s_0)(-t)e^{-s_0t}}{s-s_0}f(t)dt.$$

Using the generalized mean value theorem: $h(b)=h(a)+(b-a)h'(a)+\frac{(b-a)^2}{2}h''(\xi)$ for some $\xi\in(a,b)$, we obtain ($h(u)=e^{-ut}$, $a=s_0$, $b=s$)

$$\frac{e^{-st}-e^{-s_0t}-(s-s_0)(-t)e^{-s_0t}}{s-s_0}=t^2\frac{s-s_0}{2}e^{-\xi(s)t},\ \xi(s)\in(s_0,s).$$

Hence

$$|\frac{F(s)-F(s_0)}{s-s_0}-\mathcal{L}(-tf(t))(s_0)|\leq\frac{|s-s_0|}{2}\int_0^\infty e^{-s_0t}t^2|f(t)|dt\to 0, \tag{5.7}$$

as s tends to s_0. The integral $\int_0^\infty e^{-s_0t}t^2|f(t)|dt$ is finite if $s_0>c$, fact that goes the same way as the existence of the Laplace transform. Then passing to the limit in (5.7) ($s\to s_0$) we get $\mathcal{L}(-tf(t))(s)=F'(s),\quad s>c$. ■

Applying this theorem several time we get:

Corollary 5.4.5. *Under the same assumptions of Theorem 5.4.4, for every $n\in\mathbb{N}$, $\mathcal{L}(t^nf(t))(s)=(-1)^nF^{(n)}(s)$, $s>c$.*

Example: Problem 26, page 474. We need to calculate the inverse Laplace transform of $F(s)=\arctan\frac{3}{s+2}$. Since $F'(s)=\dfrac{-\frac{3}{(s+2)^2}}{\frac{9}{(s+2)^2}+1}=-\dfrac{3}{(s+2)^2+9}$. If f is the inverse Laplace transform of F then by Theorem 5.4.4 we get $-tf(t)=\mathcal{L}^{-1}(-\frac{3}{(s+2)^2+9})=-e^{-2t}\sin 3t$. This gives $f(t)=e^{-2t}\frac{\sin 3t}{t}$.

Another application of this formula is finding a nontrivial solution of the Bessel's type equation in Problem 34: $tx'' + (4t-2)x' + (13t-4)x = 0$, $x(0) = 0$. We denote by $X(s) = \mathcal{L}(x(t))(s)$. We have $\mathcal{L}(x'')(s) = s^2X(s) - a$ where $x'(0) = a$, and $\mathcal{L}(x')(s) = sX(s)$. Hence $\mathcal{L}(tx'') = -\frac{d}{ds}(s^2X(s) - a) = -2sX(s) - s^2X'(s)$, $\mathcal{L}(tx')(s) = -X(s) - sX'(s)$ and $\mathcal{L}(tx) = -X'(s)$.

Then the equation becomes $-2sX(s) - s^2X'(s) - 4X(s) - 4sX'(s) - 2sX(s) - 13X'(s) - 4X(s) = 0$. This reduces to a simple differential equation in $X(s)$: $\dfrac{X'(s)}{X(s)} = -\dfrac{8+4s}{(s+2)^2+9}$.

Integrating we get $\ln|X(s)| = -2\ln[(s+2)^2+9] + C$ and from here $X(s) = \frac{k}{[(s+2)^2+9]^2}$. Because $\mathcal{L}^{-1}(\frac{k}{[(s+2)^2+9]})(t) = \frac{k}{3}e^{-2t}\sin 3t$, then we can use the convolution formula to get $x(t) = \frac{k}{9}\int_0^t (e^{-2u}\sin 3u)(e^{-2(t-u)}\sin 3(t-u))du = \frac{ke^{-2t}}{18}\int_0^t \cos(6u - 3t) - \cos 3t du$. Therefore $x(t) = \frac{ke^{-2t}}{18}(\frac{\sin(6u-3t)}{6}|_0^t - t\cos 3t) = \frac{ke^{-2t}}{54}(\sin 3t - 3t\cos 3t)$ or $\boxed{x(t) = Ae^{-2t}(\sin 3t - 3t\cos 3t),\ t \geq 0}$.

Now we are going to review all the important formulae that we have introduced so far:

$f(t)$ on $[0,\infty)$	$F(s)=\mathcal{L}(f)(s)$
$\lvert f(t)\rvert \le Me^{ct}$	$\int_0^\infty e^{-st}f(t)dt,\ s>c$
$t^n,\ n\in\mathbb{N}$	$\frac{n!}{s^{n+1}},\ s>0$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2},\ s>a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2},\ s>a$
$te^{at}\cos bt$	$\frac{(s-a)^2-b^2}{[(s-a)^2+b^2]^2},\ s>a$
$te^{at}\sin bt$	$\frac{2b(s-a)}{[(s-a)^2+b^2]^2},\ s>a$
$u_a(t)$	$\frac{e^{-sa}}{s},\ s>0$
$t^a\ (a>-1)$	$\frac{\Gamma(a+1)}{s^{a+1}},\ s>0$
e^{zt}	$\frac{1}{s-z},\ s>Re\ z$
t^ne^{zt}	$\frac{n!}{(s-z)^{n+1}},\ s>Re\ z$
$e^{zt}f(t)$	$F(s-z),\ s>c+Rez$
$f'(t)$	$sF(s)-f(0),\ s>c$
$(f*g)(t)$	$F(s)G(s),\ s>c$
$tf(t)$	$-F'(s),$
$\int_0^t f(x)dx$	$F(s)/s,\ s>c$
$f(t)/t$	$\int_s^\infty F(u)du,\ s>c$
some less important	
$(-1)^{\lfloor x\rfloor}$	$\dfrac{1}{s}\tanh\dfrac{s}{2},\ s>0$
$\frac{1}{2b^3}e^{at}(\sin bt-bt\cos bt)$	$\frac{1}{[(s-a)^2+b^2]^2}$
$\frac{1}{2b}e^{at}(\sin bt+bt\cos bt)$	$\frac{(s-a)^2}{[(s-a)^2+b^2]^2}$
$e^{at}\cosh bt$	$\frac{s-a}{(s-a)^2-b^2},\ s>a$
$e^{at}\sinh bt$	$\frac{b}{(s-a)^2-b^2},\ s>a$

Homework:

Section 7.3 page 474, Problems 1-38.

5.5 Lecture XVIII

Quotation: *"When a truth is necessary, the reason for it can be found by analysis, that is, by resolving it into simpler ideas and truths until the primary ones are reached. It is this way that in mathematics speculative theorems and practical canons are reduced by analysis to definitions, axioms and postulates. " (Leibniz, 1670)*

5.5.1 Periodic and piecewise continuous input functions

We have already showed that $\mathcal{L}(u_a)(s) = \frac{e^{-as}}{s}$, $s > 0$, where

$$u_a(t) = \begin{cases} 0 \ for \ t < a \\ 1 \ for \ t \geq a. \end{cases}$$

Theorem 5.5.1. *Let us consider $a \geq 0$. If $\mathcal{L}(f)(s)$ exists for $s > c$ then*

$$\mathcal{L}(u_a(t)f(t-a))(s) = e^{-sa}\mathcal{L}(f)(s), \ \ for \ s > a.$$

PROOF. This is just a simple calculation:

$$\mathcal{L}(u_a(t)f(t))(s) = \int_0^\infty e^{-st}u_a(t)f(t-a)dt = \int_a^\infty e^{-st}f(t-a)dt =$$

$$\int_0^\infty e^{-s(u+a)}f(u)du = e^{-sa}\mathcal{L}(f)(s).$$

for all $s > c$. ■

Let us observe that if $0 < a < b$ then $u_a - u_b$ is the function:

$$u_{a,b}(t) = \begin{cases} 0 \ for \ x < a \\ 1 \ for \ x \in [a,b), \\ 0 \ for \ x \geq b. \end{cases}$$

whose graph is

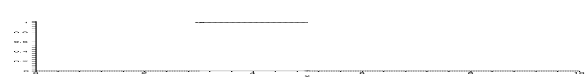


Figure 12

and it is called the characteristic function of the interval $[a, b)$. Let us solve Problem 18, page 484: we need to compute the Laplace transform of

$$f(t) = \begin{cases} \cos \frac{1}{2}\pi t & if \quad 3 \leq t \leq 5 \\ 0 \quad if & t < 3 \quad or \quad t > 5. \end{cases}$$

This function is essentially the same as $t \to u_{3,5}(t) \cos \frac{1}{2}\pi t$ for $t \in [0, \infty)$. Hence the Laplace transform of it is $\mathcal{L}(u_3(t) \cos \frac{t\pi}{2}) - \mathcal{L}(u_5(t) \cos \frac{t\pi}{2})$. Because $\cos \frac{t\pi}{2} = \cos(\frac{(t-3)\pi}{2} + \frac{3\pi}{2}) = \sin \frac{(t-3)\pi}{2}$ and similarly for $\cos \frac{t\pi}{2} = \cos(\frac{(t-5)\pi}{2} + \frac{5\pi}{2}) = -\sin \frac{(t-5)\pi}{2}$, we obtain that $\boxed{\mathcal{L}(f)(s) = (e^{-5t} + e^{-3t})\frac{2\pi}{4s^2 + \pi^2}}$.

The last theorem that we are going to do is about the transform of a periodic function:

Theorem 5.5.2. [Laplace transform of periodic function] *If f is periodic piecewise continuous with period p on $[0, \infty)$ the Laplace transfrom of f exists and*

$$\mathcal{L}(f)(s) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)dt, \ s > 0.$$

Proof. This is also a calculation:

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)dt = \sum_{k=0}^\infty \int_{kp}^{k(p+1)} e^{-st} f(t)dt$$
$$\sum_{k=0}^\infty \int_0^p e^{-st-kps} f(t + kp)dt = \sum_{k=0}^\infty e^{-kps} \int_0^p e^{-st} f(t)dt = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)dt$$

using the sum of the geometric progression $\displaystyle\sum_{k=0}^{\infty} e^{-kps} = \frac{1}{1 - e^{-sp}}$. ■

We are going to use this Theorem to compute the Laplace transform of the function in Problem 28, page 485. The graph of f (for $a = 1$ is shown below):

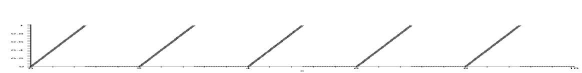


Figure 2

Basically we need to compute

$$\int_0^{2a} e^{-st} f(t)dt = \int_0^a e^{-st} t dt = e^{-st}(\frac{t}{-s} - \frac{1}{s^2})|_{t=0}^{t=a} = \frac{1 - (1 + sa)e^{-sa}}{s^2}.$$

So, the Laplace transform of f is $\boxed{\mathcal{L}(f)(s) = \dfrac{1-(1+sa)e^{-sa}}{s^2(1-e^{-2sa})}}$.

5.5.2 Impulses and delta function

Definition 5.5.3. *The Dirac delta function at a, denoted by δ_a is a transformation on continuous functions defined by $\delta_a(g) = g(a)$ for every continuous function.*

In general most of the maps φ having properties of linearity and bounded on continuous functions defined for $t \in [0,\infty)$ is of the form $\varphi(g) = \int_0^\infty g(t)h(t)dt$. The map δ_a is an example not of this form. When we have a differential equation of the type, let's say, as in Problem 2, page 495, $x'' + 4x = \delta_0 + \delta_\pi$, with initial conditions $x(0) = x'(0) = 0$, we interpret this as the model of movement of a mass ($m = 1$) attached to a spring with no dashpot with two instantaneous blows of unit intensity at moments $t = 0$ and $t = \pi$.

So if we apply both functions to the function $t \to e^{-st}$ we get $s^2X(s)+4X(s) = 1 + e^{-s\pi}$. Then we solve for $X(s) = \frac{1}{s^2+4} + \frac{e^{-s\pi}}{s^2+4}$ and then take the inverse Laplace transform $x(t) = \frac{\sin 2t}{2} + u_\pi(t)\frac{\sin 2(t-\pi)}{2}$ or $\boxed{x(t) = (1+u_\pi(t))\dfrac{\sin 2t}{2}}$

The graph of this solution is shown below. This solution is still a continuous function but it is not differentiable at every point.

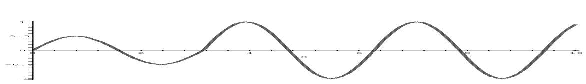


Figure 3

Now we summarize all the important Laplace transform formulae that we have studied so far:

$f(t)$ on $[0,\infty)$	$F(s)=\mathcal{L}(f)(s)$
$\lvert f(t)\rvert \le Me^{ct}$	$\int_0^\infty e^{-st}f(t)dt,\ s>c$
$t^n,\ n\in\mathbb{N}$	$\frac{n!}{s^{n+1}},\ s>0$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2},\ s>a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2},\ s>a$
$te^{at}\cos bt$	$\frac{(s-a)^2-b^2}{(s-a)^2+b^2},\ s>a$
$te^{at}\sin bt$	$\frac{2b(s-a)}{(s-a)^2+b^2},\ s>a$
$u_a(t)$	$\frac{e^{-sa}}{s},\ s>0$
$t^a\ (a>-1)$	$\frac{\Gamma(a+1)}{s^{a+1}},\ s>0$
e^{zt}	$\frac{1}{s-z},\ s>Re\ z$
t^ne^{zt}	$\frac{n!}{(s-z)^{n+1}},\ s>Re\ z$
$e^{zt}f(t)$	$F(s-z),\ s>c+Rez$
$f'(t)$	$sF(s)-f(0),\ s>c$
$(f*g)(t)$	$F(s)G(s),\ s>c$
$tf(t)$	$-F'(s),$
$\int_0^t f(x)dx$	$F(s)/s,\ s>c$
$f(t)/t$	$\int_s^\infty F(u)du,\ s>c$
some less important	
$(-1)^{\lfloor x\rfloor}$	$\frac{tanh\frac{s}{2}}{s},\ s>c$
$\frac{1}{2b^3}e^{at}(\sin bt-kt\cos bt)$	$\frac{1}{[(s-a)^2+b^2]^2}$
$\frac{1}{2b}e^{at}(\sin bt+kt\cos bt)$	$\frac{(s-a)^2}{[(s-a)^2+b^2]^2}$
$e^{at}\cosh bt$	$\frac{s-a}{(s-a)^2-b^2},\ s>a$
$e^{at}\sinh bt$	$\frac{b}{(s-a)^2-b^2},\ s>a$
$u_a(t)f(t-a)$	$e^{-sa}\mathcal{L}(f)(s)$
$f(t)$ periodic f of period p	$\frac{1}{1-e^{-sp}}\int_0^p f(t)dt$

Homework:

Section 7.5 page 484, Problems 1-35.

Section 7.6, pages 495, Problems 1-8.

Chapter 6

Power Series Methods

6.1 Lecture XIX

Quotation: *"The heart of mathematics is its problems." Paul Halmos*

6.1.1 Power series review

The method that we are going to study in this Chapter applies to a variety of DE such as the **Bessel's equation** (of order n),

$$x^2y'' + xy' + (x^2 - n^2)y = 0,$$

or Legendre's equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

which appear in many applications.

A power series around the point $x = a$ is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n(z - a)^n \tag{6.1}$$

where as usual the convergence is understood in the usual sense, i.e.

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k(z - a)^k$$

exists.

The series (2.27) defines a function $f(z)$ on a disc of radius R (called **radius of convergence**) centered at a: $D_a(R) := \{z||z-a| < R\}$.

The radius of convergence is given by the formula:

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}. \tag{6.2}$$

The series (6.1) converges at least for $z = a$ but we are going to be interested in series for which the radius of convergence is a positive real number or infinity. Most of the elementary functions have a power series expansion around any point which is not a singularity (all the derivative are defined there). The function defined by a power series is continuous and differentiable on $D_a(R)$. Moreover the derivative can be computed differentiating term by term. The derivative has the same radius of convergence and hence the function is infinitely many times differentiable. The coefficients a_n are given then by the formula:

$$a_n = \frac{f^{(n)}(a)}{n!},\ n \geq 0.$$

This allows one to compute various power series for most of the elementary functions. Two power series can be added or subtracted term by term. This corresponds to adding or subtracting the corresponding functions. The product has to be done in the Cauchy sense. The following theorem is important:

Theorem 6.1.1. **[Identity Principle]** *If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for every x in some open interval then $a_n = b_n$ for all $n \geq 0$.*

6.1.2 Series solutions around ordinary points

We are going to consider solving the DE

$$y'' + P(x)y' + Q(x)y = 0 \tag{6.3}$$

where P and Q are functions defined around point a. If these functions have a power series expansion around a then the point $x = a$ is called an **ordinary point** for (6.3). A point a will be called **singular** for (6.3) if at least one of the functions P or Q is not analytic around a (which means there is no power series centered at a that sums up to the given function). The next theorem shows what happens in the situation of ordinary points.

Theorem 6.1.2. *There are two linearly independent solutions of (6.3) around every ordinary point whose radius of convergence is at least as large as the distance from a to the nearest (real or complex) singular point of (6.3).*

Let us solve the Legendre Equation:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \tag{6.4}$$

or if we put it in the form (6.3) we get

$$y'' + \frac{-2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2}y = 0,$$

which makes it clear that $a = 0$ is an ordinary point for (6.4). According to the theorem above there are two linearly independent solutions that can be written as power series whose radius of convergence is at least 1. Let us look for a solution of the form $y(x) = \sum_{k=0}^{\infty} a_k x^k$.

Then the equation (6.4) becomes

$$(1-x^2)\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - 2x\sum_{k=0}^{\infty} a_k k x^{k-1} + \alpha(\alpha+1)\sum_{k=0}^{\infty} a_k x^k = 0$$

or

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k - 2\sum_{k=0}^{\infty} a_k k x^k + \alpha(\alpha+1)\sum_{k=0}^{\infty} a_k x^k = 0.$$

Using the identity principle we get the following equations in terms of the coefficients a_k:

$$(k+2)(k+1)a_{k+2} - k(k-1)a_k - 2ka_k + \alpha(\alpha+1)a_k = 0, \; k \geq 0.$$

This gives

$$a_{k+2} = \frac{(k-\alpha)(k+\alpha+1)}{(k+1)(k+2)}a_k, \; k \geq 0. \tag{6.5}$$

Instead of continuing this in the most general case we are going to make an assumption here that may help to see how something interesting could happen here.

Let's say $\alpha = 3$. Then (6.5) gives $a_2 = -\frac{12}{2}a_0 = -6a_0$, $a_3 = -\frac{10}{6}a_1 = -\frac{5}{3}a_1$, $a_4 = -\frac{6}{12}a_2 = 3a_0$, $a_5 = 0$. From here we see that the next coefficients $a_{2k+1} = 0$ for $k \geq 2$. So, one of the solutions is $y_1(x) = a_1(x - \frac{5}{3}x^3) = \frac{a_1}{3}(3x - 5x^3) = -\frac{2a_1}{3}P_3(x)$ where $P_3(x) = \frac{1}{2}(5x^3 - 3x)$ is the called Legendre polynomial of degree 3. Similarly for every α a non-negative integer n one of the solutions is just going to be a polynomial which turns out to be the Legendre polynomial of degree n.

Next we rewrite (6.3) as

$$y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0. \tag{6.6}$$

Definition 6.1.3. *The singular point $x = 0$ of (6.3) is a* **regular singular point** *if the functions p and q are both analytic around* 0. *Otherwise* 0 *is an* **irregular singular point**.

6.1.3 The Method of Frobenius

We are going to use a slightly modified version of power series method to solve differential equations of second order for which $x = 0$ is regular singular point. As before consider the equation written in the form

$$x^2y'' + xp(x)y' + q(x)y = 0. \tag{6.7}$$

The idea is to look simply for a solution of the form

$$x^r \sum_{k=0}^{\infty} a_k x^k, \quad x > 0. \tag{6.8}$$

We have the following theorem:

Theorem 6.1.4. *Suppose that $x = 0$ is a regular singular point for (6.7) and let $p(x) = \sum_{k=0}^{\infty} p_k x^k$ and $q(x) = \sum_{k=0}^{\infty} p_k x^k$ be the power series representations of p and q. If we denote the solutions of the quadratic equation $r(r-1) + p_0 r + q_0 = 0$ by r_1 and r_2 then:*

(a) if r_1 and r_2 are real, say $r_1 \geq r_2$, there exist a solution of the form (6.8) with $r = r_1$;

(b) if r_1 and r_2 are real, say $r_1 \geq r_2$, and $r_1 - r_2$ is not an integer (i.e. $(p_0 - 1)^2 - 4q_0$ is not the square of an integer) the there exists a second linearly independent solution of (6.7) of the form (6.8) with $r = r_2$.

Let us solve Problem 18, page 535. We need to solve the DE: $2xy''+3y'-y=0$. In this case if we write this equation in the form (6.7) we get $x^2y''+\frac{3}{2}xy'-\frac{x}{2}y=0$. This gives $x=0$ as a regular singular point and $p(x)=\frac{3}{2}$ and $q(x)=\frac{x}{2}$. Hence the equation in r becomes $r(r-1)+\frac{3}{2}r=0$ which has two solutions: $r_1=0$ and $r_2=-\frac{1}{2}$.

Therefore, according to the Theorem 6.1.4 we must have two linearly independent solutions of the form (6.8). Working out the details of this we get

$$y_1(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!(2k+1)!!},$$

and

$$y_2(x)=\frac{1}{\sqrt{x}}\left[1+\sum_{k=1}^{\infty}\frac{x^k}{k!(2k-1)!!}\right].$$

Homework:

Section 8.1, Problems pages 509-510, 23, 25 and 27;

Section 8.2, page 520, Problems 5, 6, 32, 35.

Section 8.3, page 5535, Problems 1-31, 35, 38 and 39.

6.2 Lecture XX

6.2.1 When $r_1 - r_2$ is an integer

We remind the reader the type of differential equation to which we have applied the method of Frobenius:

$$y''+\frac{p(x)}{x}y'+\frac{q(x)}{x^2}y=0. \tag{6.9}$$

where $x=0$ is a **regular singular point** of (6.11), i.e., the two functions p and q are analytic around $x=0$.

We are going to take an example from the text to study what may happen in the situation r_1-r_2 is a positive integer.

Problem 28, page 535: $xy''+2y'-4xy=0$. In this particular case $p(x)=2$ and $q(x)=-4x^2$. The equation for r (**indicitial equation**) becomes $r(r-1)+$

$2r = 0$ with solutions $r_1 = 0$ and $r_2 = -1$. This makes $r_1 - r_2$ an integer. We are going to study the existence of the second solution: $y = x^{-1}\sum_{k=0}^{\infty} a_k x^k$. Since $y' = \sum_{k=0}^{\infty}(k-1)a_k x^{k-2}$ and $y'' = \sum_{k=0}^{\infty}(k-1)(k-2)a_k x^{k-3}$, after we substitute in the given equation we get:

$$\sum_{k=0}^{\infty}(k-1)(k-2)a_k x^{k-2} + \sum_{k=0}^{\infty}2(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty}4a_k x^k = 0$$

or

$$\sum_{k=0}^{\infty}(k-1)ka_k x^{k-2} - \sum_{k=0}^{\infty}4a_k x^k = 0.$$

Since the first two terms in the first summation are zero we obtain only one summation if we shift the index $(k-2 \to k)$ and then combine the two: $\sum_{k=0}^{\infty}[(k+1)(k+2)a_{k+2} - 4a_k]x^k = 0$.

This gives $a_{k+2} = 4\dfrac{a_k}{(k+1)(k+2)}$ for all $k \geq 0$.

From here we see that $a_{2n} = \dfrac{4^n}{(2n)!}a_0$ and $a_{2n+1} = \dfrac{4^n}{(2n+1)!}a_1$ for all $n \geq 0$.

Therefore a general solution of our equation is

$$y(x) = a_0 x^{-1}\sum_{n=0}^{\infty}\frac{4^n x^{2n}}{(2n)!} + a_1\sum_{n=0}^{\infty}\frac{4^n x^{2n}}{(2n+1)!}.$$

Let us observe that we actually get an analytic solution and one which is unbounded around $x = 0$. Using the functions sinh and cosh we can re-write the general solution as

$$\boxed{y(x) = a_0 x^{-1}\cosh 2x + a_1\frac{\sinh 2x}{2x}.}$$

So, in this case we have two solutions in the form (6.8).

To show that there are cases in which there is only one solution of the form (6.8) let us take Problem 39, page 536:

(a) Show that the Bessel's equation of order 1,

$$x^2y'' + xy' + (x^2-1)y = 0$$

has exponents $r_1 = 1$ and $r_2 = -1$ at $x = 0$, and the Frobenius series corresponding to $r = 1$ is

$$J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!(n+1)!2^{2n}}.$$

(b) Show that there is no Frobenius solution corresponding to the smaller exponent $r_2 = -1$; that is, it is impossible to determine the coefficients in

$$y_2(x) = x^{-1}\sum_{n=0}^{\infty} c_n x^n. \tag{6.10}$$

Solution: Let us start by differentiating and substituting in the Bessel's equation with (6.10) as recommended (calculations will cover both cases): $y_2'(x) = \sum_{n=0}^{\infty} c_n(n-1)x^{n-2}$ and $y_2''(x) = \sum_{n=0}^{\infty} c_n(n-1)(n-2)x^{n-3}$.

The Bessel's equation becomes

$$\sum_{n=0}^{\infty} c_n(n-1)(n-2)x^{n-1} + \sum_{n=0}^{\infty} c_n(n-1)x^{n-1} + (x^2-1)\sum_{n=0}^{\infty} c_n x^{n-1} = 0.$$

The first two sums can be combined together and together with the last sum after multiplication by $x^2 - 1$ and distributing:

$$\sum_{n=0}^{\infty} c_n[n^2-2n]x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

Shifting the index of summation in the first sum we get

$$\sum_{n=-2}^{\infty} c_{n+2}(n^2+2n)x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

For $n = -2$ we get $c_0 \times 0 = 0$ which is satisfied for every c_0. For $n = -1$ we obtain $c_1 = 0$. For $n = 0$ we get $c_2 \times 0 + c_0 = 0$ which implies $c_0 = 0$. For $n \geq 1$ we have $c_{n+2} = -\frac{c_n}{n(n+2)}$. This implies $c_{2n+1} = 0$ for all $n \geq 0$ and

$$c_{2n+2} = -\frac{c_{2n}}{2n(2n+2)} = -\frac{c_{2n}}{n(n+1)2^2} = \frac{c_{2n-2}}{(n-1)nn(n+1)2^4} = ... = \frac{(-1)^n c_2}{n!(n+1)!2^{2n}},$$

for all $n \geq 1$. The final form of y_2 is

$$y_2(x) = x^{-1}\sum_{n=0}^{\infty} c_n x^n = x^{-1}\sum_{n=1}^{\infty} c_{2n}x^{2n} = x\sum_{n=1}^{\infty} c_{2n}x^{2n-2} =$$

$$= x\sum_{n=0} c^{\infty}_{2n+2}x^{2n} = c_2 x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!(n+1)!2^{2n}} = 2c_2 J_2(x).$$

This shows both parts (a) and (b) of the problem. ■

In general the equation

$$x^2y'' + xp(x)y' + q(x)y = 0 \tag{6.11}$$

has a second solution which is described by the next theorem:

Theorem 6.2.1. [**The Exceptional Case**] *Assume $x = 0$ is a regular singular point for (6.11) and $r_1 \geq r_2$ are the two roots of $r^2 + (p_0 - 1)r + q_0 = 0$.*

(a) If $r_1 = r_2$ then the equation (6.11) has two linearly independent solutions of the form:

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n, \ (a_0 \neq 0),$$

$$y_2(x) = y_1(x)\ln x + x^{r_1+1}\sum_{n=0}^{\infty} b_n x^n.$$

(b) If $r_1 - r_2 = N$ with $N \in \mathbb{N}$, then the equation (6.11) has two linearly independent solutions of the form:

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n, \ (a_0 \neq 0),$$

$$y_2(x) = Cy_1(x)\ln x + x^{r_2}\sum_{n=0}^{\infty} b_n x^n.$$

Homework:

Section 8.3, page 535, Problems 1-31, 35, 38 and 39.

Section 8.4, pages 551-552, Problems 1-8, 18, and 21.

Chapter 7

Fourier Series

7.1 Lecture XXI

Quotation: *"Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. ... A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. " George Pólya*

7.1.1 Fourier series, definition and examples

Another type of expansions for functions that can be helpful in computing solutions of differential equations is the Fourier series expansion. The method of using a different type of expantion works basically the same way as with power series: substitute in the given differential equation, find a recurrence for the coefficients and then use that to determine the coefficients and the function if possible. In general a function that has a Fourier expansion will have to be periodic. So, it is natural to work with periodic functions defined on $\mathbb{R}$ and we will take for simplicity the period to be 2π.

Definition 7.1.1. *Assume f is a piecewise continuous function of period 2π defined on $\mathbb{R}$. The Fourier series of f is*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt). \tag{7.1}$$

where $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos ntdt$ for $n=0,1,2,3,...$ and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin ntdt$ for $n=1,2,3,...$ are called the Fourier coefficients.

Let us see an example. Suppose we take Example 21, page 580. The function is $f(t) = t^2$ for $t \in [-\pi, \pi]$. Then the Fourier coefficients of f are $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt = 2\frac{\pi^3}{3\pi} = \frac{2\pi^2}{3}$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt dt = \frac{1}{\pi}\left(t^2\frac{\sin nt}{n}|_{-\pi}^{\pi} + 2t\frac{\cos nt}{n^2}|_{-\pi}^{\pi} + 2\frac{\sin nt}{n^3}|_{-\pi}^{\pi}\right)$$
$$= \frac{4(-1)^n}{n^2},$$

for $n = 1, 2, 3, ...$ and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt dt = \frac{1}{\pi}\left(-t^2\frac{\cos nt}{n}|_{-\pi}^{\pi} + 2t\frac{\sin nt}{n^2}|_{-\pi}^{\pi} + 2\frac{\cos nt}{n^3}|_{-\pi}^{\pi}\right) = 0,$$

for $n = 1, 2, 3, ...$ We will see later that this gives the following formula

$$t^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{\cos 2nt}{n^2} - 4\sum_{n=0}^{\infty}\frac{\cos(2n+1)t}{(2n+1)^2}, \tag{7.2}$$

which we will call the Fourier series expansion of f. We have to assign a meaning to the series in (7.2). As usual, we will understand by it the limit of the partial sums. If one plots the partial sums of (7.2) against $t \to t^2$ (in our plot on $[-3\pi, 3\pi]$ and taking only five terms in each sum) will get

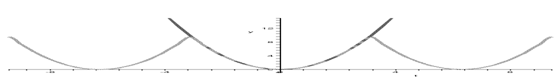


Figure 1

This suggests that the series converges to actually the given function. This usually is the case if the function is more than continuous (not true for continuous functions only) and the convergence is uniform if the function has a derivative which is piecewise continuous.

Theorem 7.1.2. [Dirichlet] *Suppose f is a periodic function of period 2π which is piecewise differentiable. The Fourier series converges*

(a) to the value of $f(t)$ for every value t where f is continuous;

(b) to the value $\frac{1}{2}(f(t+0) + f(t-0))$ at each point of discontinuity

Let us take an example where the function is discontinuous. We consider the function

$$g(t) = \begin{cases} -1 & for \quad t \in (-\pi, 0) \\ \\ 1 & for \quad t \in [0, \pi] \end{cases}$$

extended by periodicity for all real axis. Then $a_n = 0$ for all $n = 0, 1, 2, 3, 4, ...$ and $b_n = \frac{2}{\pi}\int_0^\pi \sin ntdt = \frac{2(1-(-1)^n)}{n\pi}$ for all $n = 1, 2, 3, 4, ...$

Then $g(t) = \dfrac{4}{\pi}\displaystyle\sum_{n=0}^{\infty} \frac{\sin(2n+1)t}{2n+1}$ for all $t \in (-\pi, \pi]\{0\}$. We can see that the part (b) of the Theorem 7.1.2 is satisfied. By taking $t = \frac{\pi}{2}$, we observe that this is a point of continuity for g and $g(\frac{\pi}{2}) = 1$ and hence the Theorem 7.1.2 implies that $1 = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ or

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \tag{7.3}$$

One of the important formulae that one needs in the calculation of the Fourier coefficients is given in Problem 22, page 587: *show that if $p(t)$ is a polynomial of degree n, and g is a continuous function,*

$$\int p(t)g(t)dt = p(t)G_1(t) - p'(t)G_2(t) + ... + (-1)^n p^{(n)}(t)G_{n+1}(t) + C \tag{7.4}$$

where G_{k+1} is the antiderivative of G_k for all $k = 0, 1, ..., n$ and $G_0 = g$.

This can be checked by differentiation:

$$\frac{d}{dt}(p(t)G_1(t) - p'(t)G_2(t) + ... + (-1)^n p^{(n)}(t)G_{n+1}(t)) =$$

$$p'(t)G_1(t) - p''(t)G_2(t) + ... + (-1)^n p^{(n+1)}(t)G_{n+1}(t)+$$

$$p(t)g(t) - p'(t)G_1(t) + ... + (-1)^n p^{(n)}(t)G_n(t) = p(t)g(t).$$

7.2 General Fourier Series

In general if we have a function which is periodic of period $2L$ then we can still expand it in terms of trigonometric functions but we need to change the period.

Definition 7.2.1. *If f is a piecewise continuous function of period $2L$ then the Fourier series of f is*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi t}{L}+b_n\sin\frac{n\pi t}{L}). \tag{7.5}$$

where $a_n=\frac{1}{L}\int_{-L}^{L}f(t)\cos\frac{n\pi t}{L}dt$ for $n=0,1,2,3,...$ and $b_n=\frac{1}{\pi}\int_{-L}^{L}f(t)\sin\frac{n\pi t}{L}dt$ for $n=1,2,3,...$ are called the Fourier coefficients of f on $[0,2L]$.

A similar theorem to Theorem 7.1.2 takes place in the case of periodic functions of period $2L$ ($L>0$). Let us look at the Problem 17, page 587. The function f is periodic of period 2 and defined by $f(t)=t$ for $t\in(0,2)$. We want to show that

$$f(t)=1-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi t}{n}. \tag{7.6}$$

In this cases $L=1$. Let us compute first $a_0=\int_0^2 tdt=\frac{t^2}{2}|_0^2=2$. For $n\geq 1$ we have $a_n=\int_0^2 t\cos n\pi tdt$. Using formula (7.4) we get

$$a_n=t\frac{\sin n\pi t}{n\pi}|_0^2+\frac{\cos n\pi t}{n^2\pi^2}|_0^2=0.$$

For $n\geq 1$ we have $b_n=\int_0^2 t\sin n\pi tdt$. Similarly we get

$$b_n=-t\frac{\cos n\pi t}{n\pi}|_0^2+\frac{\sin n\pi t}{n^2\pi^2}|_0^2=-\frac{2}{n\pi},$$

and so (7.6) takes place. Substituting $t=1/2$ in (7.6) will give

$$\frac{\pi}{4}=\sum_{n=0}\frac{(-1)^n}{2n+1}$$

which is nothing but the Leibniz's identity (series) (7.3).

From formula (7.2) let us derive another important series that is so common in mathematics. Denote by x the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Substituting $t=\pi$ in (7.2) we get $\pi^2=\frac{\pi^2}{3}+4x$ which will give $x=\frac{\pi^2}{6}$. Therefore

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},$$

a formula discovered by Euler. Euler and Leibnitz identities seem to be such curious facts in mathematics since they relate all the whole numbers with the number π.

Homework:

Section 9.1, page 580, Problems 1-31.

Section 9.2, pages 586-587, Problems 1-25.

Chapter 8

Miscellaneous Problems

This is a collection of problems that will help the reader to acquire an appreciation for differential equations:

Problem 1: Let u be a differentiable function defined on $\mathbb{R}$ such that

$$\frac{du}{dt} = au(t) - bu(t)^2 + h(t) \tag{8.1}$$

where h is continuous and periodic of period T on $\mathbb{R}$. Show that there are at most two T-periodic solutions of (8.1). Also, show that if there are two, they do not intersect.

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