Problems solved or proposed from the MAA Journals and other publications

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Problem 1. Let n be a positive integer, and let f be a continuous real-valued function on $[0,1]$ with the property that $\int_0^1 x^k f(x)dx = 1$ for $0 \le k \le n-1$. Prove that $\int_0^1 (f(x))^2 dx \ge n^2$.

Problem 2. Find the following limit

$$L = \lim_{n\to\infty} \frac{1}{n} \int_0^n \left(\frac{x \ln(1 + x/n)}{1+x} \right) dx. \tag{1}$$

Problem 3. Show that for $|x| < 1$,

$$\left(\sum_{n=1}^{\infty} \lfloor \frac{n}{\sqrt{2}} \rfloor x^n \right) \left(\sum_{n=1}^{\infty} x^{\lfloor n\sqrt{3} \rfloor} \right) = \left(\sum_{n=1}^{\infty} \lfloor \frac{n}{\sqrt{3}} \rfloor x^n \right) \left(\sum_{n=1}^{\infty} x^{\lfloor n\sqrt{2} \rfloor} \right), \tag{2}$$

where $\lfloor x \rfloor$ denotes the floor function.

Problem 4. For every nonnegative integer n, evaluate

$$I_n := \int_0^\infty \frac{x^n dx}{e^x + \sum_{k=0}^n \frac{x^k}{k!}}.$$