Training Putnam Problems [1]

Problems with a $\star$ are a little trickier.

1 A1 and B1 type problems

[1.] *Prove that every nonzero coefficient of the Taylor series of*

$$(1-x+x^2)e^x$$

about $x=0$ *is a rational number whose numerator (in lowest terms) is either* 1 *or a prime number.*

[2.] *For a natural number* n *we define* $R(n)=n-m^2$, *where* m *is the biggest natural number such that* $m^2\le n$. *For* $M\in\mathbb{N}$, *define the recurrent sequence* $x_1=M$, *and* $x_{k+1}=x_k+R(x_k)$ *for* $k\ge 1$. *For what values of* M, $\{x_k\}_k$ *becomes eventually constant?*

[3.] *Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them. (See Figure 1)*

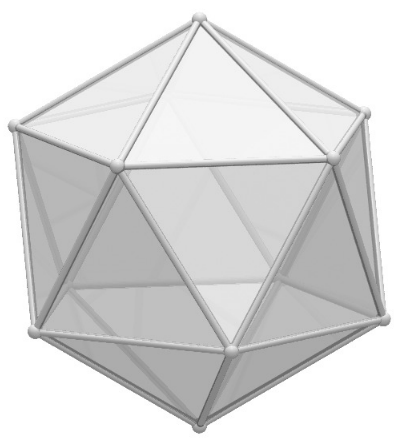

Figure 1

[4.] *A composite (positive integer) is a product* ab *with* a *and* b *not necessarily distinct integers in 2, 3, 4, ... Show that every composite number is expressible as* $xy+xz+yz+1$, *with* x, y, z, *positive integers.*

[5.] *Given a positive integer* n, *what is the largest* k *such that the numbers 1, 2, 3,...,*n *can be put into* k *boxes so that the sum of the numbers in each box is the same? [When* $n=8$, *the example* $\{1,2,3,6\}$, $\{4,8\}$, *and* $\{5,7\}$ *shows that the largest* k *is at least 3.]*

[1] Edited by Eugen J. Ionascu, last update December 3rd, 2017

6. *Let f be a real-valued function on the plane such that for every square ABCD in the plane, f(A) + f(B) + f(C) + f(D) = 0. Does it follow that f(P) = 0 for all points P in the plane?*

7. *Find polynomials f, g and h, if they exist, such that for all x*

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 \text{ if } x < -1 \\ 3x + 2 \quad \text{if } \; -1 \le x \le 0 \\ -2x + 2 \text{ if } x > 0. \end{cases}$$

8. *A right circular cone has a base of radius 1 and a height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?*

9. *Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.*

10. *Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any three (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.*

11. *Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.*

12. *Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)*

13. *A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC? (See Figure 2)*

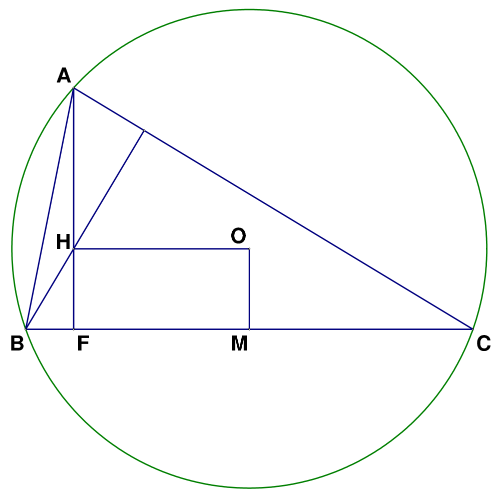

Figure 2

$\boxed{14.}$ *Consider a set S and a binary operation $\diamond$, i.e., for every $a, b \in S$, $a \diamond b \in S$. Assume $(a \diamond b) \diamond a = b$ for all $a, b \in S$. Prove that $a \diamond (b \diamond a) = b$ for all $a, b \in S$.*

$\boxed{15.}$ *Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + ... + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le ... \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways:* 4, 2+2, 1+1+2, 1+1+1+1.

$\boxed{16.}$ *Let f be a three times differentiable function on $\mathbb{R}$ having at least five distinct real zeroes. Show that*

$$f + 6f' + 12f'' + 8f'''$$

has at least two distinct real zeroes.

$\boxed{17.}$ *Find the minimum value of*

$$\frac{(x+\frac{1}{x})^6 - (x^6 + \frac{1}{x^6}) - 2}{(x+\frac{1}{x})^3 + (x^3 + \frac{1}{x^3})}$$

for $x > 0$.

$\boxed{18.}$ *Right triangle ABC has a right angle at C and $\angle BAC = t$; the pint D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = t$. The perpendicular on $\overline{BC}$ at E meets AB at F. Evaluate*

$$\lim_{t \to 0} |EF|.$$

$\boxed{19.}$ *Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, which are vertices of an equilateral triangle, and find its area.*

20. *Let f be a non-constant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.*

21. *Define an id set to be a set which has its own cardinality (number of elements) as an element. Find the number of subsets of $[m] := \{1, 2, ..., m\}$ which are minimal id sets, i.e., sets which are id sets but none of their proper subsets are id sets.*

22. *Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)*

23. *Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown. For what value of h do the rectangle and triangle have the same area?*

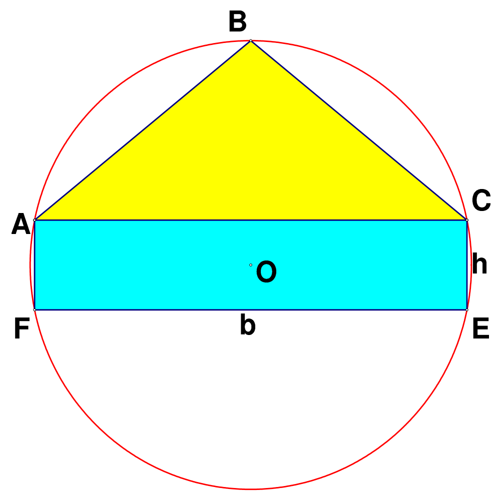

Figure 2

24. *Let A be the $n \times n$ matrix whose entry in the $i-th$ row and $j-th$ column is*

$$\frac{1}{min(i,j)}$$

for $1 \leq i, j \leq n$. Compute $det(A)$.

25. *Find the volume of the region of points (x, y, z) such that*

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

26. *Let S be a class of functions from $[0, 1)$ to $[0, 1)$ that satisfies:*
(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S;
(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;
(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

27. *What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A rational point is a point both of whose coordinates are rational numbers.)*

28. *Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.*

29. *For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of*

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

30. *Let d_1, d_2, ..., d_{12} be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, and k such that d_i, d_j, and d_k are the side lengths of an acute triangle.*

31. *Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,*

$$\frac{10}{9} = \frac{2!5!}{3!(3!)(3!)}.$$

32. *Consider all lines that meet the graph of*

$$y = 2x^4 + 7x^3 + 3x - 5$$

in four distinct points, say (x_i, y_i), $i = 1, 2, 3, 4$. Show that

$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$

is independent of the line, and find its value.

33. *Determine all polynomials $P(x)$ such that for all real values of x, we have*

$$P(x^2 + 1) = (P(x))^2 + 1 \quad \text{and } P(0) = 0.$$

34. *Evaluate the infinite product*

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}$$

35. *Let p be a prime number. Let J_p be the set of all 2×2 matrices*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose entries are chosen from the set $\{0, 1, 2, ..., p-1\}$ and which satisfy the conditions $a + d \equiv 1 \pmod{p}$ and $ad - bc \equiv 0 \pmod{p}$. Determine how many members J_p has.

36. *How many positive integers n are there such that n is an exact divisor of at least one of the numbers 10^{40} and 20^{30}?*

37. *Consider the quadratic polynomial $P(x) = ax^2 + bx + c$ with integer coefficients which has two distinct zeros in the open interval $(0, 1)$. Exhibit with a proof the least positive integer value of a for which such a polynomial P exists.*

38. *Suppose n players play a round-robin tournament (i.e., every player plays every other player exactly once.) Each game results in a win or loss for a player: there are no ties. Let w_k be the number of wins by player k, and let l_k be the number of losses by player k. Show that*

$$\sum_{k=1}^{n} w_k^2 = \sum_{k=1}^{n} l_k^2$$

39. *Let A be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7,..., 100. Prove that there must be two distinct integers in A whose sum is 104.*

40. *A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that the wrong product rule is true for x in (a, b).*

41*. *Prove that there exists a unique function $f : (0, \infty) \to (0, \infty)$ such that for all $x > 0$, we have*

$$f(f(x)) = 6x - f(x).$$

42. *For $x \in (0, 1)$, express*

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$$

as a rational function of x.

43. *Prove that among any ten consecutive integers at least one is relatively prime to each of the others.*

44. *Let a, b be given positive real numbers with $a < b$. If two points are selected at random from a straight line of length b, what is the probability that the distance between them is at least a?*

45. *Supposing that an integer n is the sum of two triangular numbers,*

$$n = \frac{a^2 + a}{2} + \frac{b^2 + b}{2}$$

write $4n + 1$ as the sum of two squares, $4n + 1 = x^2 + y^2$, and show how x and y can be expressed in terms of a and b. Show that, conversely, if $4n + 1 = x^2 + y^2$, then n is the sum of two triangular numbers.

46. *The graph of the equation $x^y = y^x$ in the first quadrant (i.e., the region where $x > 0$ and $y > 0$) consists of a straight line and a curve. Find the coordinates of the intersection point of the line and the curve.*

47. *Let $u(t)$ be a continuous function in the system of differential equations*

$$\frac{dx}{dt} = 2y + u(t), \quad \frac{dy}{dt} = 2x + u(t).$$

Show that, regardless of the choice of $u(t)$, the solution of the system which satisfies $x = x_0$, $y = y_0$ at $t = 0$ will never pass through $(0, 0)$ unless $x_0 = y_0$. When $x_0 = y_0$, show that for any positive value t_0 of t, it is possible to choose $u(t)$ so the solution is at $(0, 0)$ when $t = t_0$.

48. *What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A rational point is a point both of whose coordinates are rational numbers.)*

49. *Show that for any integers a, b, c, we can find a positive integer n such that $n^3 + an^2 + bn + c$ is not a perfect square.*

50. *What is the unit digit of*

$$\lfloor \frac{10^{20000}}{10^{100} + 3} \rfloor ?$$

51. *If n is a positive integer, demonstrate that*

$$\lfloor \sqrt{n} + \sqrt{n + 1} \rfloor = \lfloor \sqrt{4n + 2} \rfloor .$$

52. *Prove that every positive integer has a multiple whose decimal representation involves all 10 digits.*

53. *We have a hallway with* n *lockers, labeled* 1 *through* n. *The lockers have two possible states, open and closed. Initially they are all closed. The first kid walking down the hallway flips every locker to the opposite state (that is, he opens them all). The 2nd kid flips the locker door* 2 *and every other locker door after that. The* k^{th} *kid flips the state of every* k^{th} *locker door. After infinitely many kids have done this, which locker doors are closed and which are open?*

54. *Find the smallest positive integer* j *such that for every polynomial* $p(x)$ *with integer coefficients and for every integer* k, *the integer*

$$p^{(j)}(k) = \frac{d^j}{dx^j}\, p(x)|_{x=k}$$

(the j*-th derivative of* $p(x)$ *at* k*) is divisible by 2016.*

55. *Let* $x_0, x_1, x_2, \ldots$ *be the sequence such that* $x_0 = 1$ *and for* $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln *is the natural logarithm). Show that the infinite series*

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

56. *Suppose that* S *is a finite set of points in the plane such that the area of triangle* $\triangle ABC$ *is at most 1 whenever* A, B, *and* C *are in* S. *Show that there exists a triangle of area 4 that (together with its interior) covers the set* S.

57. *How many primes among the positive integers, written as usual in base* 10, *are alternating* 1*'s and* 0*'s, beginning and ending with* 1*?*

58. *Find all real-valued continuously differentiable functions* f *on the real line such that for all* x,

$$(f(x))2 = \int_0^x [(f(t))^2 + (f(t))^2]dt + 1990$$

59 *Let* L_1 *and* L_2 *be distinct lines in the plane. Prove that* L_1 *and* L_2 *intersect if and only if, for every real number* $\lambda \neq 0$ *and every point* P *not on* L_1 *or* L_2, *there exist points* A_1 *on* L_1 *and* A_2 *on* L_2 *such that* $\vec{PA_2} = \lambda \vec{PA_1}$

50 *Suppose that a positive integer* N *ca be expressed as the sum of* k *consecutive positive integers*

$$N = a + (a + 1) + (a + 2) + ... + (a + k - 1)$$

for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions ?

2 More trickier questions

1. *Let a_1, a_2, . . . , a_{2n+1} be a set of integers such that if any one of them is removed, the remaining ones can be divided into two sets of n integers with equal sums. Prove that $a_1 = a_2 = ... = a_{2n+1}$.*

2. *Let*

$$f(x) = \sum_{k=0}^{n} a_k x^{n-k}$$

be a polynomial of degree n with integral coefficients. If a_0, a_n and $f(1)$ are all odd, prove that $f(x) = 0$ has no rational roots.

3. *Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$ given that x_0, x_1, ... are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?*

4 *Let S be the smallest set of positive integers such that*
(a) 2 is in S
(b) n is in S whenever n^2 is in S, and
(c) $(n+5)^2$ is in S whenever n is in S.
Which positive integers are not in S?
(The set S is the "smallest" in the sense that S is contained in any other such set.)