Training Putnam Problems [1]

1 Inequalities

[1.] *Suppose that* x, y, $z > 0$. *Show that:*

$$\frac{x^3}{x^2+xy+y^2}+\frac{y^3}{y^2+yz+z^2}+\frac{z^3}{z^2+zx+x^2}\ge\frac{x+y+z}{3}$$

Furthermore, show that equality holds if and only if $x=y=z$.

[2.] *Given a triangle* $\triangle ABC$, *and a point* P *inside the triangle, we denote by* d_a, d_b, *and* d_c, *the distances from* P *to the lines which contain the sides of the triangle. Find the point* P *for which the product* $d_a d_b d_c$, *is maximized.*

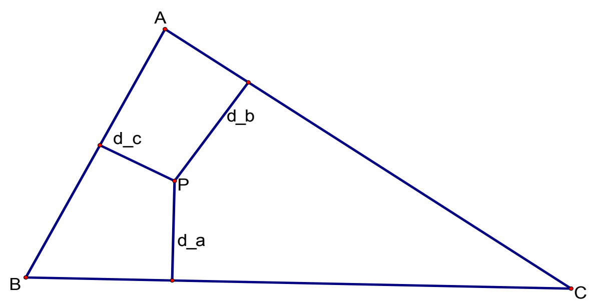


Figure 1

[3.] *Show that for every* $n\ge 2$ *(positive integer) we have*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\ge n(\sqrt[n]{n+1}-1).$$

A variation of this is the following inequality

$$n(1-\sqrt[n]{\frac{1}{2}})+\frac{1}{2n}\ge\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}\ge n(\sqrt[n]{2}-1).$$

Another variation is

$$\boxed{2n(1-\sqrt[2n]{\frac{1}{3}})+\frac{1}{3n}\ge\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n-1}\ge 2n(\sqrt[2n]{3}-1).}$$

[4.] *Prove that if* $x\ge 1$, $y\ge 1$ *and* $z\ge 1$, *then*

[1] Edited by Eugen J. Ionascu, last update January 27th, 2024

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}\geq\frac{1}{1+\sqrt{xy}}+\frac{1}{1+\sqrt{yz}}+\frac{1}{1+\sqrt{zx}}\geq\frac{3}{1+\sqrt[3]{xyz}} \tag{1}$$

5. *Let a_i, b_i be positive real numbers $i=1,2,...,n$ ($n\in\mathbb{N}$). Show that*

$$(a_1a_2\cdots a_n)^{1/n}+(b_1b_2\cdots b_n)^{1/n}\leq\left[(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)\right]^{1/n}. \tag{2}$$

6. *Let x, y, and z be positive numbers such that $x+y+z=3$. Prove that*

$$\frac{x^4+x^2+1}{x^2+x+1}+\frac{y^4+y^2+1}{y^2+y+1}+\frac{z^4+z^2+1}{z^2+z+1}\geq 3xyz \tag{3}$$

7. *For positive real numbers a, b and c, show that*

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2} \text{ Nasbitt's Inequality.}$$

A variation of this is for four numbers a, b, c and d:

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\geq 2.$$

8. *Suppose that a_1, a_2, ..., a_n with $n\geq 2$ are real numbers greater that -1, and all the numbers a_j have the same sign. Show that*

$$(1+a_1)(1+a_2)\cdots(1+a_n)>1+a_1+a_2+\cdots+a_n.$$

As a corollary, we have Bernoulli's inequality

$$(1+x)^n\geq 1+nx,\quad n\geq 1, x>-1,$$

but it is a calculus exercise to show it works for $n\geq 1$.

9. *Show that if a, b and c are positive then*

$$\ln\frac{27abc}{(a+b+c)^3}\leq\frac{(a-b)^2+(b-c)^2+(a-c)^2}{3}.$$

10. *Let $n\in\mathbb{N}$ and x_1, x_2, ..., x_n be n real numbers greater or equal than 1 such that $\sum_{i=1}^{n}\frac{1}{x_i}=1$. Prove that*

$$\frac{n}{1/2+n^2}\leq\sum_{i=1}^{n}\frac{1}{1/2+x_i^2}\leq\frac{2}{3}. \tag{4}$$

$\boxed{11.}$ [B2-2005] *Find all positive integers n, k_1, k_2, ..., k_n such that*

$$k_1 + \cdots + k_n = 5n - 4 \quad \textit{and} \quad \frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1$$

$\boxed{12.}$ [B2 1988] *Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.* **Hint:** See the figure below!

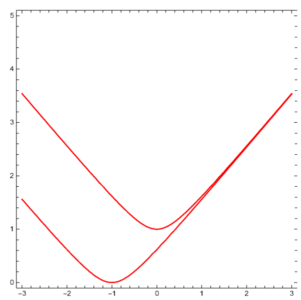


Figure 2

$\boxed{13.}$ [**Hardy's Inequality**] (a) Suppose that $p < 1$ and we have a sequence of non-negative numbers $\{a_n\}$. The we have

$$\sum_{n=1}^{\infty} \left(\frac{a_1^p + a_2^p + \cdots + a_n^p}{n} \right)^{1/p} \leq q^{1/p} \sum_{n=1}^{\infty} a_n,$$

where $q = 1/(1-p)$. Show that if $p = 1$ the left hand side in the above inequality may not be convergent while the right hand side is a convergent series. The expression in the summand in the left hand side is called the p-**power mean** of a_1, a_2, ...,a_n:

$$M_p(a_1, a_2, ..., a_n) = \left(\frac{a_1^p + a_2^p + \cdots + a_n^p}{n} \right)^{1/p}.$$

(see Power mean)

(b) The continuous version is that for a non-negative p-integrable function f (here $p > 1$), we have

$$\int_0^{\infty} (\frac{1}{x} \int_0^x f(t) dt)^p dx \leq (\frac{p}{p-1})^p \int_0^{\infty} f(x)^p dx.$$

Hint: Substitution $f(x) = g(x^{1-\frac{1}{p}}) x^{-\frac{1}{p}}$ ... and then Jensen's Inequality.

(c) [**Carleman's Inequality**] If $\{a_n\}$ is a sequence of positive numbers, then

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 a_2 \cdots a_n} \leq e \sum_{n=1}^{\infty} a_n.$$

(d) [B2- 2021] The problem was formulated in a different but equivalent way:

$$\sum_{n=1}^{\infty} \frac{n}{2^n}(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{2}{3} \sum_{k=1}^{\infty} a_k.$$

(e) If $\{a_n\}$ is a sequence of positive numbers, then

$$\sum_{n=1}^{\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq 2 \sum_{n=1}^{\infty} a_n.$$

(f) If $\{a_n\}$ is a sequence of positive numbers, then

$$\sum_{n=1}^{\infty} \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq \sum_{n=1}^{\infty} a_n.$$

14. **Problem 4857. Crux 49(6) (June Issue 2023)** Let a, b, c be positive real numbers such that $a + b + c = \frac{3}{2}$. Show that $a^a b^b + b^b c^c + c^c a^a \geq \frac{3}{2}$.

15. **A strange inequality** Show that for all $x, y, z \geq 0$, we have

$$\frac{x^3 + 2}{2 + x + y + z^3} + \frac{y^3 + 2}{2 + y + z + x^3} + \frac{z^3 + 2}{2 + z + x + y^3} \geq \frac{9}{5}. \tag{5}$$

2 Discrete Math

1. *Given n a non-negative integer, find the largest power of 2 which divides $\lfloor (1 + \sqrt{3})^{2n+1} \rfloor$. Here $\lfloor x \rfloor$ denotes the largest integer which is less than equal to x.*

2. *Prove that there exists a unique function f from the set $(0, \infty)$ of positive real numbers to $(0, \infty)$ such that*

$$f(f(x)) = 6x - f(x)$$

and $f(x) > 0$ for all $x > 0$.

3. *Let n be a positive even integer. We write the numbers $1, 2, \cdots n^2$ in a square grid such that the k-th row, from left to right reads:*

$$(k-1)n + 1, (k-1)n + 2, \cdots, (k-1)n + n$$

We then color the squares on the grid so that half of the squares in each row and in each column are colored red and half are colored blue. Show that the sum of the numbers which are colored red is equal to the sum of the numbers which are colored blue.

4. *Suppose that n is a positive integer. Prove that:*

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

3 Geometry

1. *Suppose that $ABCD$ is a tetrahedron in 3-space (which is not necessarily regular). At each face S_j; $j = 1, ..., 4$ of the tetrahedron, we draw a vector $\vec{n}_j$ which satisfies:*

i) $\vec{n}_j$ is perpendicular to S_j,

ii) $\vec{n}_j$ points outwards,

iii) $|\vec{n}_j|$ equals the surface area of S_j.

Show that: $\vec{n}_1 + \vec{n}_2 + \vec{n}_3 + \vec{n}_4 = \vec{0}$.

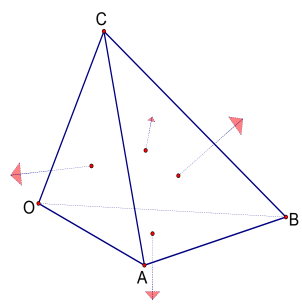


Figure 2

2. *The graph below is the polar curve of*

$$r = \sqrt{\cos 2\theta} + \frac{7}{5}\cos\theta$$

in the Cartesian plane for values of θ where $\sqrt{\cos 2\theta}$ exists. Determine with proof whether the graph is a circle.

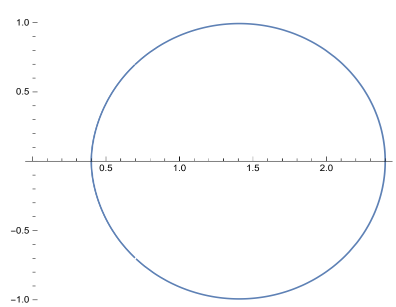


Figure 3

3. *The figure below is a trapezoid ($\overline{DC}||\overline{AE}$) with $CD = AD = AB = BE$ and $\angle DAB = 90°$. Cut this region inside of the trapezoid into four congruent regions.*

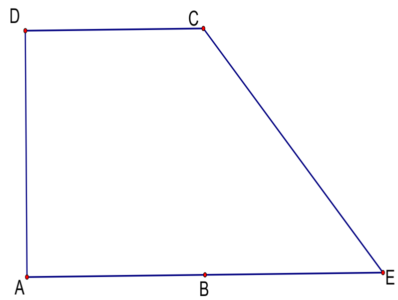


Figure 4

4. *Triangle ABC has area 1. Points E, F, G lie, respectively, on sides BC, CA AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.*

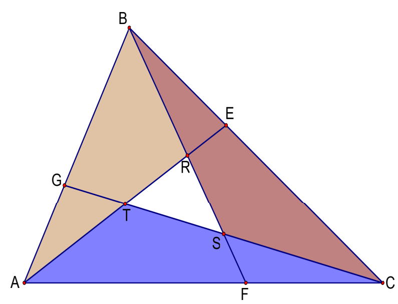


Figure 5

5. Let a, b, and c be the side lengths of a nondegenerate, nonequilateral triangle with largest angle α. Let T be the set of lengths t such that there exists an equilateral triangle ABC in the plane with origin O such that $AB = t$, $OA = c$, $OB = a$, and $OC = b$.

(a) Prove that $|T| = 2$.

(b) Prove that the smaller of the two equilateral triangles determined by T does not contain O in its interior.

(c) Prove that the larger of the two equilateral triangles determined by Tcontains O in its interior if and only if $\frac{\pi}{3} < \alpha < \frac{2\pi}{3}$.

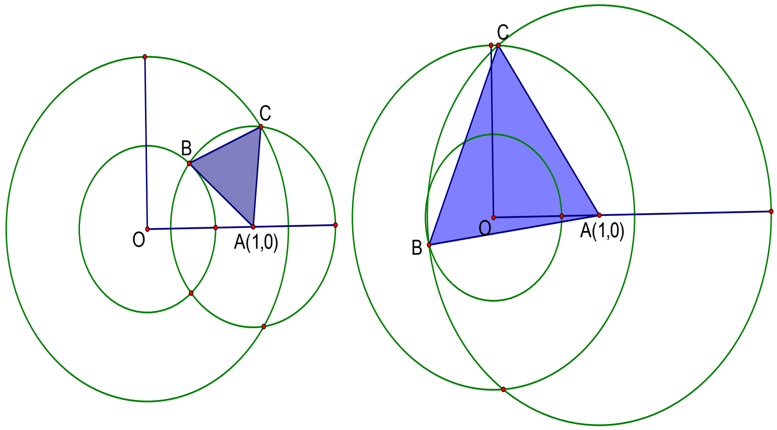


Figure 6, $\triangle ABC$

6. [A-1 (2009)] *Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D)=0$. Does it follows that $f(P)=0$ for all points in the plane?*

7. [B-1 1986]

Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown. For what value of h do the rectangle and triangle have the same area?

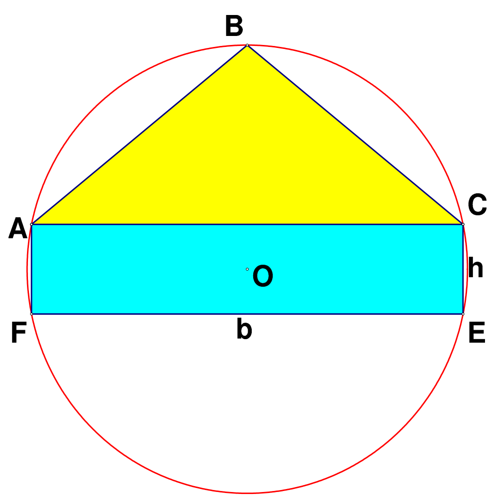


Figure 7

7. [B-4 1985]

Let C be the unit circle $x^2+y^2=1$. A point p is chosen randomly on the circumference of C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x and y-axes with diagonal pq. What is the probability that no point of R lies outside of C ?

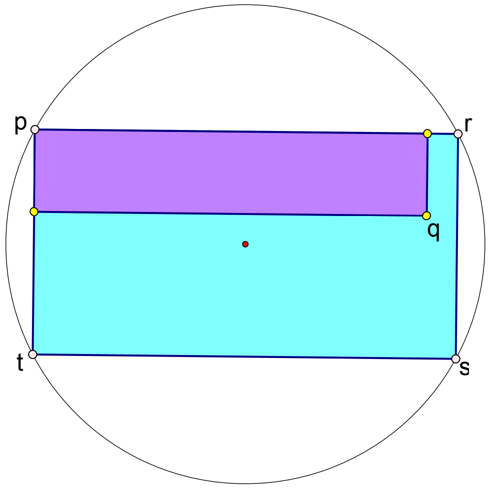


Figure 8

$\boxed{8.}$ Given a right triangle ABC with $m(\angle ACB) = 90^\circ$ and $BC = 2AC$, construct isosceles right triangles $\triangle ABF$ and $\triangle ACE$ on AC and AB as in Figure 2 (right angles at E and F, both points in the interior of the anlge $\angle BAC$). Show that EF is parallel to AC.

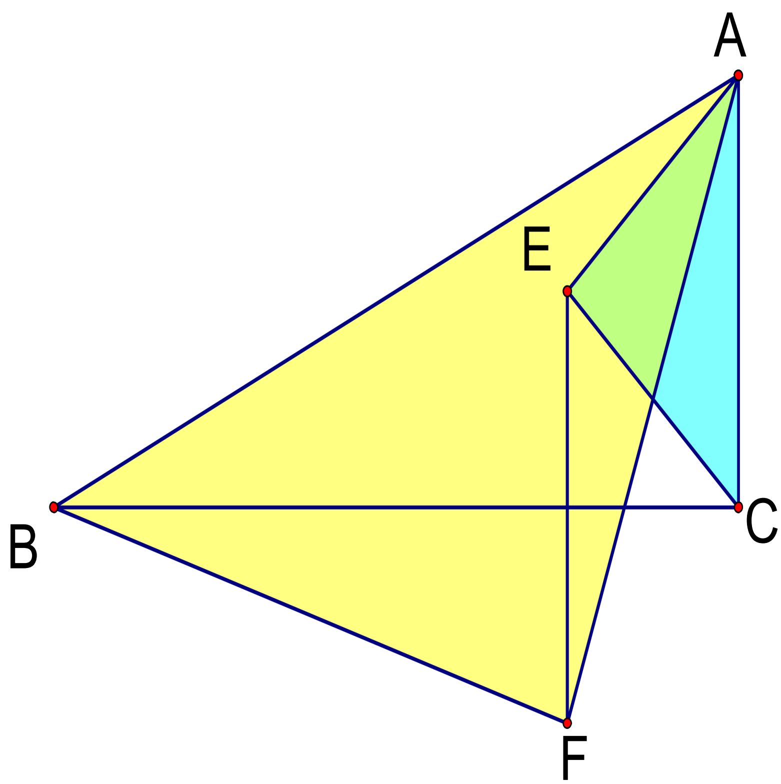


Figure 2, Right triangle at C, BC=2AC

4 Number Theory

Some facts about 2022:

- $2023 = 7 \cdot 17 \cdot 17$
- $2023 = 2^3 + (-9)^3 + 14^3$, $7 = 0^3 + (-1)^3 + 2^3$, and $17 = 1^3 + 2^3 + 2^3$.
- Sum of all divisors $\sigma(n) = 2456$
- There are 1,632 positive integers (up to 2023) that are relatively prime to 2023. In other "words" $\phi(2023) = 6 \cdot 17 \cdot 16 = 1632$.
- $2023 = 2^{11} - 5^2$

1. *Suppose that P is a polynomial with integer coefficients and suppose that there exists a positive integer n such that none of the values $P(1)$, $P(2)$, ... , $P(n)$ are divisible by n. Show that P doesn't have any integer roots.*

2. *Show that all of the numbers F_0, F_1, ..., F_n where $F_k = 2^{2^k} + 1$ (Fermat's numbers), are pairwise relatively prime.*

Remark: It is known that the first five terms of $\{F_k\}$ are primes and it is not known if there are any other primes in this sequence:

$$\{F_0, F_1, ...\} = \{3, 5, 17, 257, 65537, 4294967297, 18446744073709551617, ...\}$$

Fermat wrongly conjectured that all of these numbers are primes and Euler proved that $F_5 = 641 \cdot 6700417$. We notice that this problem shows there are infinitely many primes.

3. *Suppose that p is a prime number. Show that $(p-1)! + 1$ is divisible by p.*

4. *Let us denote by A the set of all positive integers which are not divisible by the square of any prime number, i.e.,*

$$A = \{n \in \mathbb{N} |\ q \in \mathbb{N}, q^2 | n^2 \implies q = 1\} = \{1, 2, 3, 5, 6, 7, 10, ...\}$$

Given a positive integer n, show that:

$$\sum_{k \in A} \lfloor \sqrt{n/k} \rfloor = n.$$

Hint: *Induction on n, and look for the terms in the sum that are changing their values.*

5. *Suppose p is a prime number. Prove that*

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \quad (mod\ p^2).$$

Hint: *(Recall that $\sum_{j=0}^{p} \binom{p}{j} = 2^p$.)*

6. *A positive integer is* **squarely correct** *if it is a perfect square or if its base-10 representation consists entirely of adjacent blocks of digits that are* **positive** *perfect squares. For example,* 99 *and* 100 *are two consecutive numbers that are both squarely correct. However,* 101 *is not squarely correct- only positive perfect squares are allowed by definition.*

a) Are there infinitely many pairs of consecutive correct numbers ?

b) It is possible to find three or more consecutive squarely correct number ?

Remark: The sequence of squarely correct numbers in increasing order starts like this :

$$1, 4, 9, 11, 14, 16, 19, 25, 36, 41, 44, 49, 64, 81, 91, 94, 99, 100, ...$$

(check to see if I missed any of them !)

7. *A heronian triangle is a triangle with positive integer side lengths and positive integer area. Denoting the side lengths of a Heronian triangle by a, b and c, the triangle is called* **primitive** *if* $\gcd(a, b, c) = 1$. *We shall say that a primitive Heronian triangle has* **an equivalent rectangle** *if there exists a rectangle with integer length and width that shares the same perimeter and area as the triangle. Show that infinitely many primitive Heronian triangles have equivalent rectangles.*

8. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

9. [A1 -2005] Show that every integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example $23 = 9+8+6$.)

10. [B-1 1988] A composite (positive integer) is a product ab with a and b not necessarily distinct integers in $\{, 2, 3, 4, ...\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, and z positive integers.

5 Real Analysis and differential equations

1. *Suppose that f is a non-negative continuous function on $\mathbb{R}$. Suppose that, for every $\epsilon \in [0,1)$, one has $lim_{n\to\infty} f(\epsilon+n) = 0$. Show by example, that $\lim_{x\to\infty} f(x)$ doesn't have to equal zero.*

For the next exercise, let us first recall the following notation. Given a real number x, let $\lfloor x \rfloor$ equal the largest integer which is less than or equal to x. For instance, $\lfloor 2.5 \rfloor = 2$, and $\lfloor -3.7 \rfloor = -4$. The quantity $\lfloor x \rfloor$ is called the floor of x. We also define the fractional part of x by $\{x\} := x - \lfloor x \rfloor$ We note that $\{x\} \in [0,1)$.

2. *Suppose that α is a real number.*
a) If α is rational, show that the set

$$X_\alpha := \{\{nx\} | n \in \mathbb{Z}\}$$

is not dense in $[0,1)$.
b) If α is irrational, show that the set X_α, defined as above, is dense in $[0,1)$.

3. *Does there exist an integer n such that the number 2^n in the decimal system starts with the digits 2022?*

4. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that:*

$$f''(x) + f(x) = -xg(x)f'(x)$$

for some non-negative function $g : \mathbb{R} \to \mathbb{R}$. Show that the function f is bounded.

6. *It is easy to show that if the series $\sum a_j$ is convergent then $\{a_j\}$ is convergent to zero. Show that there exists a sequence $\{a_j\}$ not convergent to zero, such that*

$$\sum_{\lfloor \frac{n}{2} \rfloor + 1}^{n} a_j$$

is convergent as $n \to \infty$.

7. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function satisfying $xf(y) + yf(x) \leq 1$ for every $x, y \in [0,1]$*
(a) Show that $\int_0^1 f(x)dx \leq \frac{\pi}{4}$.
(b) Find such a function for which equality occurs.

8. *(a) For what pairs of positive real numbers (a,b) does the improper integral shown below converges?*

$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

(b) *For $a = b = 1$, show that*

$$\int_b^\infty \left(\sqrt{\sqrt{x}-\sqrt{x-1}} - \sqrt{\sqrt{x+1}-\sqrt{x}}\right) dx = \frac{4}{15}(\sqrt{26\sqrt{2}-14}-2)$$

9. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1$, $f'(0) = 0$ and for all $x \in [0, \infty)$, it satisfies*

$$f''(x) - 5f'(x) + 6f(x) \geq 0.$$

Prove that, for all $x \in [0, \infty)$,

$$f(x) \geq 3e^{2x} - 2e^{3x}.$$

10. *Let $y = f(x)$ be a continuous, strictly increasing function of x for $x \geq 0$, with $f(0) = 0$, and f^{-1} denote the inverse function of f. If a and b are nonnegative constants, then show that*

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy.$$

11. *[B5-1985] For $a > 0$ calculate the integral*

$$\int_0^\infty x^{-\frac{1}{2}} e^{-a(x+\frac{1}{x})} dx$$

You may assume that $\int_{-\infty}^{\infty} e^{x^2} dx = \sqrt{\pi}$.

12. *Let f be continuous function defined on $[0, 1]$. Prove that*

$$\int_0^1 \int_0^1 |f(x) + f(y)| dx dy \geq \int_0^1 |f(x)| dx.$$

13. *For every positive integer n, set $a_n := \sum_{k=1}^n \frac{1}{k^4}$ and $b_n = \sum_{k=1}^n \frac{1}{2k-1)4}$. Compute*

$$\lim_{n\to\infty} n^3 (\frac{b_n}{a_n} - \frac{15}{16})$$

14. *Do there exist functions $f : (0, 1) \to \mathbb{R}$ and $g : (0, 1) \to \mathbb{R}$ such that for all x and $y \in (0, 1)$, the following two conditions are satisfied:*
   1. *$f(x) < g(x)$, and*
   2. *if $x < y$, then $g(x) < f(y)$?*

   Either find examples of such f and g or prove that no such functions exist. 14. *Find the volume of the ellipsoid*

$$x^2 + y^2 + z^2 + xy + yz + zx = 1.$$

$\boxed{15.}$ *[A-2 (2009)] Functions f, g, h are differentiable on some interval around 0 and satisfy the equations and initial conditions:*

$$f' = 2f^2gh + \frac{1}{gh}, \quad f(0) = 1,$$

$$g' = f^2gh + \frac{1}{fh}, \quad g(0) = 1,$$

$$h' = 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1.$$

Find an explicit formula for $f(x)$, valid in some open interval around 0. **Hint:** *Get and equation only in terms of $\frac{f'}{f}$, $\frac{g'}{g}$, and $\frac{h'}{h}$.*

$\boxed{16.}$ [Problem 12340 (AMM 2022 August-September)] `Let` $g : [0,1] \to \mathbb{R}$ `be a continuous function.`

$$\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n} dx = Cg(1/2). \tag{6}$$

`for some constant` C `(independent of` g`) and determine the value of` C.

$\boxed{17.}$ [A-5 2005] *Evaluate*

$$\int_0^1 \frac{\ln(x+1)}{1+x^2} dx.$$

$\boxed{18.}$ [B-1 1987] Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}} dx.$$

$\boxed{19.}$ [A-1 1986] Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying

$$x^4 + 36 \leq 13x^2.$$

$\boxed{20.}$ [A-2 1986] Evaluate $\sum_{n=0}^{\infty} Arccot(n^2+n+1)$, where $Arccot\, t$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \frac{\pi}{2}$ with $\cot\theta = t$.

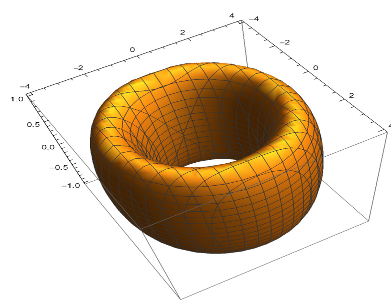
$\boxed{21.}$ [A-2 1988] A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

$\boxed{22.}$ [B-1 1990] Find all real valued continuously differentiable functions f on the real line such that for all x,

$$f(x)^2 = \int_0^x [f(t)^2 + f'(t)^2]dt + 1990.$$

$\boxed{23.}$ [A-1 2006] Find the volume of the region $\mathcal{R}$ of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$



Region $\mathcal{R}$ (figure not provided on the test)

$\boxed{24.}$ [A-1 2000] Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that x_0, x_1, ... are positive numbers for which $\sum_{j=0} x_j = A$?

$\boxed{25.}$ [B-6 2006] Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

$\boxed{26.}$ Let $f(x) = \sum_{n=1}^{\infty} \frac{|\sin nx|}{n^2}$. Prove that

$$\lim_{x\to 0^+} \frac{f(x)}{x \ln x} = -1. \tag{7}$$

27. Let r be a positive real number. Evaluate

$$I := \int_0^\infty \frac{x^{r-1}}{(1+x^2)(1+x^{2r})} dx. \tag{8}$$

28. Evaluate

$$L := \lim_{n\to\infty} \frac{\ln n^{1/3} \ln(n+3)}{\sum_{1\le i<j\le n} \frac{1}{ij}}. \tag{9}$$

29. Evaluate

$$I := \int_0^1 \frac{\sin x \sin \pi x}{\cos \frac{2x-1}{2}} dx. \tag{10}$$

30. Let $H_n = \sum_{k=1}^n \frac{1}{k}$. Evaluate

$$S := \sum_{n=1}^\infty \frac{H_{n+2}}{n(n+1)}. \tag{11}$$

6 Functional Equations

1. *(i) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that:*

$$f(x+y) = f(x) + f(y)$$

for every x and y real numbers. Show that $f(x) = cx$ for some constant c.

(ii) If the function satisfies instead

$$f(x+y) = f(x)f(y)$$

for every x and y real numbers, show that $f(x) = e^{cx}$ for some constant c, or $f \equiv 0$.

(iii) Suppose $f : (0,\infty) \to \mathbb{R}$ is a differentiable function such that:

$$f(xy) = f(x) + f(y)$$

for every x and y positive real numbers. Show that $f(x) = c \ln x$ for some constant c.

(iv) Suppose $f : (0,\infty) \to \mathbb{R}$ is a differentiable function such that:

$$f(xy) = f(x)f(y)$$

for every x and y positive real numbers. Show that $f(x) = x^c$ for some constant c.

(v) The conclusion in all statements above follows if f is only assumed continuous.

2. *Suppose that f and g are two real-valued functions defined on the whole real line, such that for all x and y*

$$f(x+y) = f(x)f(y) - g(x)g(y) \quad \textit{and}$$

$$g(x+y) = f(x)g(y) + g(x)f(y).$$

Knowing that $f'(0) = 0$ show that $f(x)^2 + g(x) = 1$ for all x.

3. Find all functions $: (0,\infty) \to (0,\infty)$ such that

$$f(kx + f(y)) = \frac{y}{k} f(xy+1) \tag{12}$$

for all x and y in $(0,\infty)$, where $k > 0$ is a real and fixed parameter.

4. [A1 1992] *Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that:*

(i) $f(f(k)) = k$,

(ii) $f(f(k+2)+2) = k$, and

(iii)$f(0) = 1$

for all integers k.

5. [B3-2005] *Find all differentiable functions $f : (0,\infty) \to (0,\infty)$ for which there is a positive real number a such that*

$$f'(\frac{a}{x}) = \frac{x}{f(x)}$$

for all $x > 0$.

6. **Crux** *Proposed by Ivan Hadinata.* Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equation

$$f(xy + f(f(y)) = xf(y) + y \tag{13}$$

holds for all real numbers x and y.

7. **Monthly** For fixed $p \in \mathbb{R}$, find al function $f : [0,1] \to \mathbb{R}$ that are continuous at 0 and 1 and satisfy

$$f(x^2) + 2pf(x) = (x+p)^2 \tag{14}$$

for all $x \in [0,1]$.

7 Algebra

1. *Find the minimum value of the expression:*

$$\frac{(x+\frac{1}{x})^6-(x^6+\frac{1}{x^6})-2}{(x+\frac{1}{x})^3+(x^3+\frac{1}{x^3})}$$

for $x>0$.

2. *(i) Prove the Lagrange's identities:*

$$(a^2+b^2)(u^2+v^2)=(au+bv)^2+(av-bu)^2$$

$$(a^2+b^2+c^2)(u^2+v^2+w^2)=(au+bv+cw)^2+(av-bu)^2+(aw-cu)^2+(bw-cv)^2$$

and more general

$$(\sum_{k=1}^n a_k^2)(\sum_{k=1}^n b_k^2)=(\sum_{k=1}^n a_kb_k)^2+\sum_{i<j}(a_ib_j-a_jb_i)^2.$$

(ii) Prove Cauchy's Inequality:

$$(\sum_{k=1}^n a_k^2)(\sum_{k=1}^n b_k^2)\geq(\sum_{k=1}^n a_kb_k)^2.$$

(iii) Show that the Diophantine equation $x^2y^2+y^2z^2+z^2x^2=w^2$ $(x,y,z,w\in\mathbb{Z})$ *has infinitely many solutions. Hint: Use the identity*

$$(ab)^2+(ab+a)^2+(ab+b)^2=(a^2+ab+b^2)^2.$$

4. *Let* G *be a finite group, and suppose that for any subgroups* H *and* K *of* G*, we have*

$$|H\cup K|=\gcd(|H|,|K|).$$

Prove that G *is cyclic.*

5. *Prove that if*

$$11z^{10}+10iz^9+10iz-11=0$$

then $|z|=1$. *(Here* z *is a complex number and* $i^2=-1$.*)* **Hint:** *Show that there are* 10 *roots on the circle already.*

6. A **Gaussian integer** is a complex number z such that $z=a=bi$ for integers a and b. Show that every Gaussian integer can be written in at most one way as

a sum of distinct powers of $1+i$, and that the Gaussian integer z can be expressed as such a sum if and only if $i-z$ cannot.

$\boxed{7.}$ Find a nonzero polynomial $P(x,y)$ such that $P(\lfloor a \rfloor, (\lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor x \rfloor$ is the greatest integr lass than or equal to x.)

$\boxed{8.}$ [B-2 1986] Prove that there are only a finite number of possibilities for the ordered triple $T = (x-y, y-z, z-x)$, where x, y and Z are complex numbers satisfying the simultaneous equations

$$x(x-1) + 2yz = y(y-1) + 2zx = z(z-1) + 2xy,$$

and list all such triples T.

$\boxed{9.}$ [A-2 1986] What is the units digit of

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor ?$$

$\boxed{10.}$ [A-1 1987] Curves A, B, C and D are defined in the plane as follows:

$$A = \{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\},$$

$$B = \{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\},$$

$$C = \{(x,y) : x^3 - 3xy^2 + 3y = 1\},$$

$$D = \{(x,y) : 3x^2y - 3x - y^3 = 0\}.$$

Prove that $A \cap B = C \cap D$.

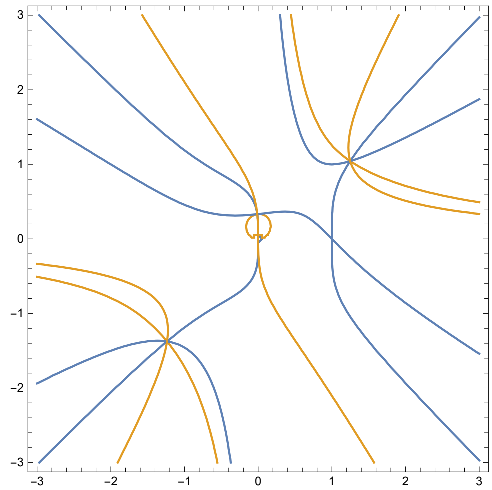


Figure 9

$\boxed{11.}$ [B-3 1987] Let F be a field in which $1+1 \neq 0$. Show that the set of solutions to the equation $x^2+y^2=1$ with x and y in F is given by $(x,y)=(1,0)$ and

$$(x,y)=\left(\frac{r^2-1}{r^2+1}, \frac{2r}{r^2+1}\right)$$

where r runs through the elements of F such that $r^2 \neq -1$.
Bonus: Show that the map above is $1-1$ and calculate how many pairs are in $\mathbb{Z}_p$ with p prime.

$\boxed{12.}$ [A-1 1988] Let R be the region consisting of the points (x,y) of the cartesian plane satisfying

$$|x|-|y| \leq 1 \quad \text{and} \quad |y| \leq 1.$$

Sketch the region R and find its area.

$\boxed{13.}$ [B-2 1989] Let S be a non-empty set with an associative operation that is left and right cancellation ($xy=xz$ implies $y=z$ and yzx implies $y=z$). Assume that for every a in S the set $\{a^n : \ n=1,2,3,...\}$ is finite. Must S be a group ?

$\boxed{14.}$ Show the identities

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-xz-yz),$$

$$x^2+y^2+z^2-xy-xz-yz=\frac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2].$$

15. [B1-2005] Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.

16. [A1-2019] Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B and C are non-negative integers.

8 Set Theory

1. *Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite ?*