79^{th} Putnam Mathematical Competition, December 1^{th}, 2018

Morning Session

$\boxed{A1.}$ Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

$\boxed{A2.}$ Let S_1, S_2, ..., S_{2^n-1} ne the nonempty subsets of $\{1, 2, ..., n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if} \quad S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

$\boxed{A3.}$ Determine the greatest possible value of $\sum_{i=1}^{10} \cos(3x_i)$ for real numbers x_1, x_2, ..., x_{10} satisfying $\sum_{i=1}^{10} \cos(x_i) = 0$.

$\boxed{A4.}$ Let m and n be positive integers with $gdc(m, n) = 1$, and let

$$a_k = \lfloor \frac{mk}{n} \rfloor - \lfloor \frac{m(k-1)}{n} \rfloor$$

for $k = 1, 2, ..., n$. Suppose that g and h are elements in a group G and that

$$gh^{a_1} gh^{a_2} ... gh^{a_n} = e,$$

where e is the identity element. Show that $gh = hg$. (As usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

$\boxed{A5.}$ Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$ and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

$\boxed{A6.}$ Suppose A, B, C, and D are distinct points, no three of which lie on a line, in the Euclidean plane. Show that if the squares of the lengths of the line segments AB, AC, AD, BC, BD, and CD are rational numbers, then the quotient

$$\frac{area(\triangle ABC)}{area(\triangle ABD)}$$

is a rational number.

Afternoon Session

$\boxed{B1.}$ Let $\mathcal{P}$ be the set of vectors defined by

$$\mathcal{P} = \left\{ \left(\begin{array}{c} a \\ b \end{array} \right) \middle| \; 0 \leq a \leq 2, \; 0 \leq b \leq 100, \quad a, b \in \mathbb{Z} \right\}.$$

Find all $\mathbf{v} \in \mathcal{P}$ such that the set $\mathcal{P} \backslash \{\mathbf{v}\}$ obtained by omitting vector $\mathbf{v}$ from $\mathcal{P}$ can be partitioned into two sets of equal size and equal sum.

$\boxed{B2.}$ Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + ... + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

$\boxed{B3.}$ Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.

$\boxed{B4.}$ Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and

$$x_{n+1} = 2x_n x_{n-1} - x_{n-2}$$

for $n \geq 2$. prove that if $x_n = 0$ for some n, then the sequence is periodic.

$\boxed{B5.}$ Let $f = (f_1, f_2)$ be a function from $\mathbb{R}^2 \to \mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{1}{4}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\right)^2 > 0$$

everywhere. Prove that f is one-to-one.

$\boxed{B6.}$ Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$